J. math. fluid mech. 8 (2006) 147–186
1422-6928/06/020147-40

DOI 10.1007/s00021-004-0141-7

Journal of Mathematical Fluid Mechanics

Leading Order Down-Stream Asymptotics of Stationary Navier–Stokes Flows in Three Dimensions

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Communicated by H. Amann

Abstract. We consider stationary solutions of the incompressible Navier–Stokes equations in three dimensions. We give a detailed description of the fluid flow in a half-space through the construction of an inertial manifold for the dynamical system that one obtains when using the coordinate along the flow as a time.

Mathematics Subject Classification (2000). 35Q30, 76D05, 35B40, 35B42.

Keywords. Navier–Stokes equations, asymptotics, down-stream, laminar wake.

1. Introduction

We consider, in $d = 3$ dimensions, the time independent incompressible Navier–Stokes equations

$$-(\mathbf{u}\cdot\nabla)\mathbf{u} + \Delta\mathbf{u} - \nabla p = \mathbf{0}, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

in a half-space $\Omega = \{(x,\mathbf{y}) \in \mathbf{R}^3 \mid x \geq 1\}$. We are interested in modeling the situation where fluid enters the half-space Ω through the surface $\Sigma = \{(x,\mathbf{y}) \in \mathbf{R}^3 \mid x = 1\}$ and where the fluid flows at infinity parallel to the x-axis at a nonzero constant speed $\mathbf{u}_\infty \equiv (1,\mathbf{0})$. We therefore impose the boundary conditions

$$\lim_{\substack{x^2+|\mathbf{y}|^2\to\infty \\ x\geq 1}} \mathbf{u}(x,\mathbf{y}) = \mathbf{u}_\infty, \tag{3}$$

$$\mathbf{u}|_\Sigma = \mathbf{u}_\infty + \mathbf{u}_*, \tag{4}$$

with $\mathbf{u}_*$ in a certain set of vector fields satisfying $\lim_{|\mathbf{y}|\to\infty}\mathbf{u}_*(\mathbf{y}) = \mathbf{0}$.

The following theorem is our main result.

*Supported in part by the Fonds National Suisse.

Theorem 1. *Let Σ and Ω be as defined above. Then, for each $\mathbf{u}_* = (u_*, \mathbf{v}_*)$ in a certain set of vector fields $\mathcal{S}$ to be defined later on, there exist a vector field $\mathbf{u} = \mathbf{u}_\infty + (u, \mathbf{v})$ and a function p satisfying the Navier–Stokes equations* (1) *and* (2) *in Ω subject to the boundary conditions* (3) *and* (4). *Furthermore,*

$$\lim_{x\to\infty} x \left(\sup_{\mathbf{y}\in\mathbf{R}^2} |(u - u_{as})(x,\mathbf{y})| \right) = 0, \tag{5}$$

$$\lim_{x\to\infty} x^{3/2} \left(\sup_{\mathbf{y}\in\mathbf{R}^2} |(\mathbf{v}_1 - \mathbf{v}_{1,as})(x,\mathbf{y})| \right) = 0, \tag{6}$$

$$\lim_{x\to\infty} x \left(\sup_{\mathbf{y}\in\mathbf{R}^2} |(\mathbf{v}_2 - \mathbf{v}_{2,as})(x,\mathbf{y})| \right) = 0, \tag{7}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the irrotational and divergence free parts of $\mathbf{v}$, respectively, where

$$u_{as}(x,\mathbf{y}) = \frac{1}{4\pi x} e^{-\frac{y^2}{4x}}\ c + \frac{1}{2\pi}\frac{x}{r^3}\ d + \frac{1}{2\pi}\frac{\mathbf{y}\cdot\mathbf{b}}{r^3}, \tag{8}$$

$$\begin{aligned} \mathbf{v}_{1,as}(x,\mathbf{y}) &= \frac{\mathbf{y}}{8\pi x^2} e^{-\frac{y^2}{4x}}\ c + \frac{1}{2\pi}\frac{\mathbf{y}}{r^3}\ d \\ &\quad - \frac{1}{2\pi}\frac{1}{r}\frac{\operatorname{sign}(x)}{r+|x|}\left(\mathbf{1} - \frac{1}{r}\left(\frac{1}{r} + \frac{1}{r+|x|}\right)\mathbf{y}\mathbf{y}^T\right)\mathbf{b}, \end{aligned} \tag{9}$$

$$\begin{aligned} \mathbf{v}_{2,as}(x,\mathbf{y}) &= \frac{1}{4\pi x} e^{-\frac{y^2}{4x}}\ \mathbf{a} \\ &\quad + \frac{1}{2\pi}\left(\frac{1}{y^2}\left(e^{-\frac{y^2}{4x}} - 1\right)\mathbf{1} - 2\frac{1}{y^4}\left(e^{-\frac{y^2}{4x}} - 1 + \frac{y^2}{4x}e^{-\frac{y^2}{4x}}\right)\mathbf{y}\mathbf{y}^T\right)\mathbf{a}, \end{aligned} \tag{10}$$

with $y = \sqrt{y_1^2 + y_2^2}$, where $(y_1, y_2) = \mathbf{y}$, with $r = \sqrt{x^2 + y^2}$, with $\mathbf{1}$ the unit 2×2 matrix, with $\mathbf{y}\mathbf{y}^T$ the 2×2 matrix with entries $\left(\mathbf{y}\mathbf{y}^T\right)_{ij} = y_i y_j$, and where the numbers c and d and the vectors $\mathbf{a}$ and $\mathbf{b}$ are related to the initial conditions u_* and $\mathbf{v}_*$ as follows,

$$d = \left\langle -i\mathbf{e}^T \lim_{k\to 0} \widehat{\mathbf{v}_{*,1}}(k\mathbf{e}) \right\rangle, \tag{11}$$

$$c = -d + \left\langle \lim_{k\to 0} \widehat{u_*}(k\mathbf{e}) \right\rangle, \tag{12}$$

$$\mathbf{b} = \left\langle -i\mathbf{e} \lim_{k\to 0} \widehat{u_*}(k\mathbf{e}) \right\rangle, \tag{13}$$

$$\mathbf{a} = -\mathbf{b} + \left\langle 2 \lim_{k\to 0} \widehat{\mathbf{v}_{*,2}}(k\mathbf{e}) \right\rangle - \left\langle 2 \lim_{k\to 0} \widehat{\mathbf{v}_{*,1}}(k\mathbf{e}) \right\rangle, \tag{14}$$

where $\widehat{\ }$ denotes Fourier transform, where $\mathbf{e} \equiv \mathbf{e}(\vartheta) = (\cos(\vartheta), \sin(\vartheta))$ and where $\mathbf{v}_{*,1}$ and $\mathbf{v}_{*,2}$ are the irrotational and divergence free parts of $\mathbf{v}_*$, respectively. The *average $\langle\ .\ \rangle$ is defined by the equation*

$$\langle\ .\ \rangle = \frac{1}{2\pi}\int_0^{2\pi}\ .\ \ d\vartheta. \tag{15}$$

A proof of this theorem is given in Section 9.

The set $\mathcal{S}$ in Theorem 1 will be specified in Section 9, once appropriate function spaces have been introduced. For related results see for example [6], [7], [4], [8], [12], [5] and references therein. For an application of analogous two-dimensional results for a numerical implementation of two-dimensional stationary exterior flow problems see [2] and [3].

Theorem 1 has the following interpretation: consider a rigid body $\mathbf{B}$ (a compact set with smooth boundary) of diameter L that is placed into a uniform stream of a homogeneous incompressible fluid, filling up all of $\mathbf{R}^3$. Experimentally, far away from the body, such a fluid flow appears to be close to a potential flow with the exception of a region downstream of the object, the so called wake region, within which the vorticity of the fluid is concentrated. This situation is modeled by the equations

$$-\rho\ (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} + \mu\Delta\tilde{\mathbf{u}} - \nabla\tilde{p} = 0, \tag{16}$$

$$\nabla\cdot\tilde{\mathbf{u}} = 0, \tag{17}$$

in $\tilde{\Omega} = \mathbf{R}^3\setminus\mathbf{B}$, subject to the boundary condition $\tilde{\mathbf{u}}|_{\partial\tilde{\Omega}} = \mathbf{0}$, $\lim_{|\tilde{\mathbf{x}}|\to\infty}\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = \tilde{\mathbf{u}}_\infty = (u_\infty, 0)$. If we assume that the density ρ and the dynamic viscosity μ of the fluid are constant in $\tilde{\Omega}$, then we can always choose a coordinate system as indicated in Figure 1, scale to dimensionless coordinates $\mathbf{x} = (\rho u_\infty/\mu)\,\tilde{\mathbf{x}}$, introduce a dimensionless vector field $\mathbf{u}$ and a dimensionless pressure p by defining $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = u_\infty\mathbf{u}(\mathbf{x})$ and $\tilde{p}(\tilde{\mathbf{x}}) = \left(\rho u_\infty^2\right)p(\mathbf{x})$. In the new coordinates equations (16), (17) become equal to (1), (2) with Σ located at $x = 1$.

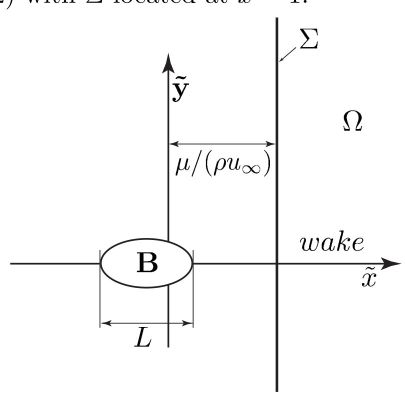

FIG. 1. Stationary flow around a body $\mathbf{B}$

For solutions $\tilde{\mathbf{u}}$ of (16) which are such that the corresponding scaled vector field $(\mathbf{u}-\mathbf{u}_\infty)|_\Sigma \in \mathcal{S}$ (we expect this to be all solutions of (16) for which the Reynolds number $\mathrm{Re} = L\rho u_\infty/\mu$ is small enough, but we do not address this question here), Theorem 1 shows in particular the existence of a parabolic wake, within which the leading order deviation from the constant flow is universal, *i.e.*, independent of the details of the shape of the body. On a heuristic level this is a well known fact [1]. It is related to what is called a "change of type" of equation (16) from an elliptic partial differential equation to a parabolic partial differential equation. The mathematical tools that we use to prove this change of type are a version of the center manifold theorem as proved in [9], combined with asymptotic expansion techniques. See also [14], [15] for a similar interpretation of the analogous two-dimensional results.

The present results generalize the techniques that we introduced in [10] from two to three dimensions. This generalization is in many ways straightforward and the basic techniques applied here are close to the ones used in [10], but new difficulties arise do to the presence of a branch of zero-modes for the velocity field and the nontrivial algebraic structure of the equations for the velocity and vorticity components transverse to the flow.

We finally note that, based on rigorous results proved in [7] and on theoretical arguments and numerical results presented in [2] and [3] (see also [11]), we expect (8)–(10) to be the dominant asymptotic behavior not only for $x \to \infty$, but on all curves for which $|x| + |y| \to \infty$, provided that we replace in (8)–(10) c by $-2d\theta(x)$ and $\mathbf{a}$ by $-2\mathbf{b}\theta(x)$ with θ the Heaviside step function ($\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$). We address this question briefly in Section 7 and in Appendix VI.

The rest of this paper is organized as follows. In Section 2, Section 3 and Section 4 we rewrite equation (1), (2) as a dynamical system with the coordinate parallel to the flow playing the role of time. The discussion will be formal. At the end of the discussion we get a set of integral equations. In Sections 5 and 6 we then prove that these integral equations admit a solution. This solution is analyzed in detail in Section 7 and Section 8. In Section 9 we finally prove Theorem 1 by using the results from Sections 5–8.

2. The dynamical system

Define, for given $\mathbf{u}$ and p, the vector field $\mathbf{U}$ by the equation

$$\mathbf{U} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + \Delta\mathbf{u} - \nabla p. \tag{18}$$

With this notation, the Navier–Stokes equations (1), (2) are

$$\mathbf{U} = \mathbf{0}, \tag{19}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{20}$$

Let $\mathbf{W} = \nabla \times \mathbf{U}$ be the vorticity of the vector field $\mathbf{U}$. After some vector algebra (or see *e.g.* [1]), we find from (18) that

$$\mathbf{W} = \nabla\times(\mathbf{u} \times \omega) + \Delta\omega, \tag{21}$$

with

$$\omega = \nabla \times \mathbf{u} \tag{22}$$

the vorticity of the fluid. Note that

$$\nabla \cdot \omega = 0, \tag{23}$$

$$\nabla \cdot \mathbf{W} = 0. \tag{24}$$

The Navier–Stokes equation (20) can be solved by first determining ω and $\mathbf{u}$ from the vorticity equations

$$\mathbf{W} = \mathbf{0}, \tag{25}$$

together with (22) and (20), and then the pressure p by solving the Poisson equation

$$\Delta p = \nabla\cdot((\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta\mathbf{u})$$

in Ω, subject to the boundary condition $\mathbf{U} = \mathbf{0}$ normal to the boundary Σ.

As in [10] we consider now the coordinate parallel to the flow as a time coordinate, and rewrite the equations (25) as a dynamical system. We first introduce some notation. Let $\mathbf{x} = (x, \mathbf{y})$ with $\mathbf{y} = (y_1, y_2)$, $\mathbf{u} = (1, \mathbf{0}) + (u, \mathbf{v})$ with $\mathbf{v} = (v_1, v_2)$, $\omega = (w, \tau)$ with $\tau = (\tau_1, \tau_2)$, $\nabla = (\partial_x, \nabla')$ with $\nabla' = (\partial_{y_1}, \partial_{y_2})$, $\Delta' = \partial_{y_1}^2 + \partial_{y_2}^2$, and let $\mathbf{W} = (W, \mathbf{T})$. Furthermore, $(\nabla')^{\perp} = (-\partial_{y_2}, \partial_{y_1})$, $\mathbf{v}^{\perp} = (-v_2, v_1)$, $\tau^{\perp} = (-\tau_2, \tau_1)$ and similarly for other vectors in $\mathbf{R}^2$. Then, we find for ω as defined in (22)

$$\omega = \begin{pmatrix} -\nabla' \cdot \mathbf{v}^{\perp} \\ (\partial_x \mathbf{v})^{\perp} - (\nabla')^{\perp} u \end{pmatrix}, \tag{26}$$

and similarly,

$$\mathbf{u} \times \omega = \begin{pmatrix} -\mathbf{v} \cdot \tau^{\perp} \\ \tau^{\perp} + (u\tau - \omega\mathbf{v})^{\perp} \end{pmatrix}. \tag{27}$$

Therefore,

$$\nabla\times(\mathbf{u} \times \omega) = \begin{pmatrix} \nabla' \cdot \tau \\ -\partial_x \tau \end{pmatrix} + \begin{pmatrix} \nabla' \cdot \mathbf{q}_0 \\ -\mathbf{q} \end{pmatrix}, \tag{28}$$

where

$$\mathbf{q} = \partial_x \mathbf{q}_0 + (\nabla')^{\perp} q_1, \tag{29}$$

and

$$\mathbf{q}_0 = u\tau - \omega\mathbf{v}, \tag{30}$$

$$q_1 = -\mathbf{v} \cdot \tau^{\perp}. \tag{31}$$

Using (26) and (28), we find that the equations (25), (22) and (20) are in component form equal to

$$\omega = -\nabla' \cdot \mathbf{v}^{\perp}, \tag{32}$$

$$\tau = (\partial_x \mathbf{v})^{\perp} - (\nabla')^{\perp} u, \tag{33}$$

$$0 = \partial_x u + \nabla' \cdot \mathbf{v}, \tag{34}$$

$$W = \nabla' \cdot \tau + \nabla' \cdot \mathbf{q}_0 + \partial_x^2 \omega + \Delta' \omega = 0, \tag{35}$$

$$\mathbf{T} = -\partial_x \tau - \mathbf{q} + \partial_x^2 \tau + \Delta' \tau = \mathbf{0}. \tag{36}$$

Equations (33), (34) and (36) are equivalent to

$$\partial_x \tau = \eta, \tag{37}$$

$$\partial_x \eta = \eta - \Delta' \tau + \mathbf{q}, \tag{38}$$

$$\partial_x u = -\nabla' \cdot \mathbf{v}, \tag{39}$$

$$\partial_x \mathbf{v} = \nabla' u - \tau^{\perp}. \tag{40}$$

Equations (37)–(40) are very similar to the equations (16) in [10], and are the dynamical system which we will study below.

The remaining two equations (32) and (35) have no two-dimensional analogue. They are related to the fact that the vector fields ω and $\mathbf{W}$ have to be divergence free.

Equation (32) provides an explicit way of computing the first component ω of the vorticity in terms of the transverse components $\mathbf{v}$ of the velocity field, at any "time" x. We will also need expressions for the "time"-derivatives $\partial_x \omega$ and $\partial_x^2 \omega$ of ω. Differentiating (32) with respect to x and using (40) shows that

$$\partial_x \omega = -\nabla' \cdot \tau, \tag{41}$$

which is nothing else than $\nabla \cdot \omega = 0$ written in component form, and differentiating (41) with respect to x and using (37) leads to

$$\partial_x^2 \omega = -\nabla' \cdot \eta. \tag{42}$$

We now discuss equation (35). To motivate what follows we first note that on the linear level the *r.h.s.* in (37)–(40) depends only on the irrotational part of $\mathbf{v}$. This is the reason for the appearance of a branch of zero modes in our dynamical system (see below), and is related to the fact that W is a (nonlinear) invariant of the dynamical system. Namely, differentiating (35) with respect to x and using (37)–(40) and (32) we find that

$$\partial_x W = -\nabla' \mathbf{T}, \tag{43}$$

which is nothing else than $\nabla \cdot \mathbf{W} = 0$ written in component form. From (43) it follows that $\partial_x W = 0$ if $\mathbf{T} = 0$, which means that the function W is independent of x if the equations (37)–(40) and (32) are satisfied, and it is therefore sufficient to require W to be zero at $x = 1$ (or any other convenient value of x) in order to satisfy equation (35). Therefore, and provided we can rewrite W in such a way that it does not contain derivatives with respect to x anymore, equation (35) can be solved by choosing appropriate "initial conditions" for our dynamical system. Indeed, using (42) we get from (35) that

$$W = \Delta'\omega + \nabla' \cdot (\tau - \eta) + \nabla' \cdot \mathbf{q}_0, \tag{44}$$

with ω given by (32), and this is an expression for W containing only the "dynamical variables" τ, η, u, and $\mathbf{v}$, and is free of x-derivatives.

We conclude that the system of equations (32)–(36) is equivalent to the dynamical system (37)–(40) with the nonlinear term $\mathbf{q}$ as defined in (29)–(31), with ω as defined in (32) and with initial conditions chosen such that W as defined in (44) equals zero.

3. Fourier transforms

Following the ideas in [10], we now Fourier transform equation (37)–(40) in the transverse directions. Throughout this and subsequent sections, vectors will be treated notation-wise as 2×1 matrices, an upper script ^{T} meaning matrix transposition. Let

$$\tau(x, \mathbf{y}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbf{R}^2} e^{-i\mathbf{k}\cdot\mathbf{y}} \ \hat{\tau}(\mathbf{k}, x) \ d^2\mathbf{k},$$

with $\mathbf{k} = (k_1, k_2)$, and accordingly for the other functions. For (37)–(40) we then get (for simplicity we drop the hats and use in Fourier space t instead of x for the "time"-variable),

$$\begin{aligned} \dot{\tau} &= \eta, \\ \dot{\eta} &= \eta + k^2\tau + \mathbf{q}, \\ \dot{u} &= i\mathbf{k}^T\mathbf{v}, \\ \dot{\mathbf{v}} &= -i\mathbf{k}u - \tau^{\perp}, \end{aligned} \tag{45}$$

the dot meaning derivative with respect to t and $k = \sqrt{k_1^2 + k_2^2}$. From (29), (30) and (31) we get that

$$\mathbf{q} = \partial_t \mathbf{q}_0 - i\mathbf{k}^{\perp} q_1, \tag{46}$$

where

$$\mathbf{q}_0 = \frac{1}{4\pi^2}(u * \tau - \omega * \mathbf{v}), \tag{47}$$

$$q_1 = -\frac{1}{4\pi^2}\mathbf{v}^T * \tau^{\perp}, \tag{48}$$

and $*$ being the convolution product. Equation (32) becomes

$$\omega = i\mathbf{k}^T\mathbf{v}^{\perp}, \tag{49}$$

and from (44) we find that

$$W = -k^2\omega - i\mathbf{k}^T(\tau - \eta) - i\mathbf{k}^T\mathbf{q}_0. \tag{50}$$

3.1. Stable and unstable modes

The equations (45) are of the form $\dot{\mathbf{z}} = L\mathbf{z} + \chi$, with $\mathbf{z} = (\tau, \eta, u, \mathbf{v})$, $\chi = (\mathbf{0}, \mathbf{q}, 0, \mathbf{0})$ and with L a matrix with the following block structure

$$L = \begin{pmatrix} L_1 & 0 \\ L_3 & L_2 \end{pmatrix}, \tag{51}$$

with L_1 a 4×4 matrix L_2 a 3×3 matrix, L_3 a 3×4 matrix and 0 the 4×3 zero matrix. For L_1 we have

$$L_1(\mathbf{k}) = \begin{pmatrix} 0 & 1 \\ k^2 & 1 \end{pmatrix}, \tag{52}$$

the matrix entries being 2×2 matrices. For L_2 we have

$$L_2(\mathbf{k}) = \begin{pmatrix} 0 & i\mathbf{k}^T \\ -i\mathbf{k} & 0 \end{pmatrix}, \tag{53}$$

the first line of the matrix consisting of a 1×1 and a 1×2 matrix and the second line of a 2×1 and a 2×2 matrix, and for L_3 we have

$$L_3(\mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{54}$$

the last column of the matrix consisting of 1×2 matrices. The matrix $L(\mathbf{k})$ can be diagonalized (see Appendix I for details). Namely, let

$$\Lambda_0(k) = \sqrt{1 + 4k^2}, \qquad \Lambda_+(k) = \frac{1 + \Lambda_0(k)}{2}, \qquad \Lambda_-(k) = \frac{1 - \Lambda_0(k)}{2},$$

and let $\mathbf{z} = S\zeta$, with S a matrix with the same block structure as L,

$$S = \begin{pmatrix} S_1 & 0 \\ S_3 & S_2 \end{pmatrix}, \tag{55}$$

with

$$S_1(\mathbf{k}) = \begin{pmatrix} 1 & 1 \\ \Lambda_+ & \Lambda_- \end{pmatrix}, \qquad S_2(\mathbf{k}) = \begin{pmatrix} 0 & 1 & 1 \\ \frac{i}{k}\mathbf{k}^\perp & -\frac{i}{k}\mathbf{k} & \frac{i}{k}\mathbf{k} \end{pmatrix}, \tag{56}$$

$$S_3(\mathbf{k}) = \begin{pmatrix} \frac{i}{\Lambda_+}\left(\mathbf{k}^\perp\right)^T & \frac{i}{\Lambda_-}\left(\mathbf{k}^\perp\right)^T \\ \frac{-1}{\Lambda_+}\frac{\mathbf{k}^\perp\mathbf{k}^T}{k^2} + \frac{\mathbf{k}\left(\mathbf{k}^\perp\right)^T}{k^2} & \frac{-1}{\Lambda_-}\frac{\mathbf{k}^\perp\mathbf{k}^T}{k^2} + \frac{\mathbf{k}\left(\mathbf{k}^\perp\right)^T}{k^2} \end{pmatrix}. \tag{57}$$

Then, $\dot{\zeta} = D\zeta + S^{-1}\chi$, with

$$S^{-1} = \begin{pmatrix} S_1^{-1} & 0 \\ \left(S^{-1}\right)_3 & S_2^{-1} \end{pmatrix} \tag{58}$$

again a matrix with the same block structure as L, with

$$S_1^{-1}(\mathbf{k}) = \begin{pmatrix} -\frac{\Lambda_-}{\Lambda_0} & \frac{1}{\Lambda_0} \\ \frac{\Lambda_+}{\Lambda_0} & -\frac{1}{\Lambda_0} \end{pmatrix}, \qquad S_2^{-1}(\mathbf{k}) = \begin{pmatrix} 0 & \frac{-i}{k}\left(\mathbf{k}^\perp\right)^T \\ \frac{1}{2} & \frac{i}{2k}\mathbf{k}^T \\ \frac{1}{2} & \frac{-i}{2k}\mathbf{k}^T \end{pmatrix}, \tag{59}$$

$$\left(S^{-1}\right)_3(\mathbf{k}) = \begin{pmatrix} \frac{i}{k^3}\mathbf{k}^T & -\frac{i}{k^3}\mathbf{k}^T \\ (k-1)\frac{-i}{2k^2}\left(\mathbf{k}^\perp\right)^T & \frac{-i}{2k^2}\left(\mathbf{k}^\perp\right)^T \\ (k+1)\frac{-i}{2k^2}\left(\mathbf{k}^\perp\right)^T & \frac{-i}{2k^2}\left(\mathbf{k}^\perp\right)^T \end{pmatrix}, \tag{60}$$

and with $D = S^{-1}LS$ a diagonal matrix with diagonal entries Λ_+, Λ_+, Λ_-, Λ_-, 0, k, and $-k$ (see Figure 2). Note that $\Lambda_+(k) \geq 1$ and $\Lambda_-(k) \leq 0$ and that $\Lambda_-(k) \approx -k^2$ for small values of k. We also have the identities $\Lambda_+ + \Lambda_- = 1$, $\Lambda_+ - \Lambda_- = \Lambda_0$, and $\Lambda_+\Lambda_- = -k^2$, which will be routinely used below.

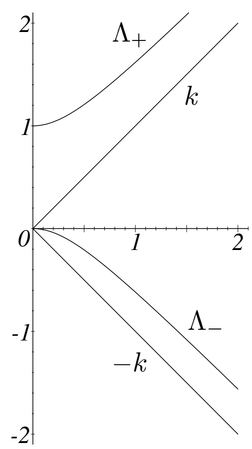

FIG. 2. The spectral branches Λ_+, Λ_-, k and $-k$

Let $\zeta = (\tau_+, \tau_-, v_0, v_+, v_-)$. Using the definitions we find that the equations (37)–(40) are equivalent to

$$\dot{\tau}_+ = \Lambda_+\tau_+ + \frac{1}{\Lambda_0}\mathbf{q},$$
$$\dot{\tau}_- = \Lambda_-\tau_- - \frac{1}{\Lambda_0}\mathbf{q},$$

$$
\begin{aligned}
\dot{v}_0 &= \frac{-i}{k^3}\mathbf{k}^T\mathbf{q}, \\
\dot{v}_+ &= k\ v_+ + \frac{1}{2k^2} i\mathbf{k}^T\mathbf{q}^\perp, \\
\dot{v}_- &= -k\ v_- + \frac{1}{2k^2} i\mathbf{k}^T\mathbf{q}^\perp,
\end{aligned} \tag{61}
$$

with $\mathbf{q}$ as defined in (46)–(48). For convenience later on we write $\mathbf{z} = S\zeta$ in component form. Namely,

$$\tau = \tau_+ + \tau_-, \tag{62}$$

$$\eta = \Lambda_+\tau_+ + \Lambda_-\tau_-, \tag{63}$$

$$u = -i\mathbf{k}^T\bar{\tau}^\perp + v_+ + v_-, \tag{64}$$

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \tag{65}$$

where

$$\bar{\tau} = \frac{1}{\Lambda_+}\tau_+ + \frac{1}{\Lambda_-}\tau_-, \tag{66}$$

$$\mathbf{v}_1 = -P_1\tau^\perp - \frac{i}{k}\mathbf{k}\ (v_+ - v_-), \tag{67}$$

$$\mathbf{v}_2 = -P_2\bar{\tau}^\perp + \frac{i}{k}\mathbf{k}^\perp\ v_0, \tag{68}$$

and where

$$P_1(\mathbf{k}) = \frac{\mathbf{k}\mathbf{k}^T}{k^2}, \qquad P_2(\mathbf{k}) = \frac{\mathbf{k}^\perp\left(\mathbf{k}^\perp\right)^T}{k^2},$$

are the projection operators on the irrotational and divergence free part of a vector field, respectively. Next, using that $\mathbf{k}^T P_1 = \mathbf{k}^T$ and $P_1\mathbf{v}^\perp = (P_2\mathbf{v})^\perp$, we get from (49) the following expression for ω,

$$\omega = i\mathbf{k}^T P_1\mathbf{v}^\perp = i\mathbf{k}^T\left(P_2\mathbf{v}\right)^\perp = i\mathbf{k}^T\left(\mathbf{v}_2\right)^\perp. \tag{69}$$

Using (68), and furthermore that $\left(-P_2\bar{\tau}^\perp\right)^\perp = P_1\bar{\tau}$ and that $\left(\mathbf{k}^\perp\right)^\perp = -\mathbf{k}$, we get that

$$\omega = i\mathbf{k}^T\bar{\tau} + k\ v_0, \tag{70}$$

and therefore we find from (50) using (62), (63) and the identity $\Lambda_+ + \Lambda_- = 1$ that

$$W = -k^3 v_0 - i\mathbf{k}^T\mathbf{q}_0. \tag{71}$$

We conclude that the system of equations (32)–(36) is equivalent to the dynamical system (61) with the nonlinear term $\mathbf{q}$ as defined in (46), (47), (48), with ω as defined in (69), with τ, η, u, $\mathbf{v}$ as defined in (62), (63), (64) and (65), and with initial conditions chosen such that W as defined in (71) equals zero.

4. The integral equations

To solve (61) we convert it into an integral equation. The +-modes are unstable (remember that $\Lambda_+(k) \geq 1$) and we therefore have to integrate these modes backwards in time starting with $\tau_+(\mathbf{k},\infty) \equiv u_+(\mathbf{k},\infty) \equiv 0$. We get

$$\tau_+(\mathbf{k},t) = -\frac{1}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)}\mathbf{q}(\mathbf{k},s)\ ds, \tag{72}$$

$$\tau_-(\mathbf{k},t) = \tilde{\tau}_-^*(\mathbf{k})e^{\Lambda_-(t-1)} - \frac{1}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)}\mathbf{q}(\mathbf{k},s)\ ds, \tag{73}$$

$$v_0(\mathbf{k},t) = v_0^*(\mathbf{k}) + \frac{i}{k^3}\mathbf{k}^T\int_t^\infty \mathbf{q}(\mathbf{k},s)\ ds, \tag{74}$$

$$v_+(\mathbf{k},t) = \frac{-i}{2k^2}\mathbf{k}^T\int_t^\infty e^{k(t-s)}\mathbf{q}(\mathbf{k},s)^\perp\ ds, \tag{75}$$

$$v_-(\mathbf{k},t) = \tilde{v}_-^*(\mathbf{k})e^{-k(t-1)} + \frac{i}{2k^2}\mathbf{k}^T\int_1^t e^{-k(t-s)}\mathbf{q}(\mathbf{k},s)^\perp\ ds. \tag{76}$$

Equation (46) implies that $\mathbf{k}^T\mathbf{q}(\mathbf{k},t) = \partial_t\mathbf{k}^T\mathbf{q}_0(\mathbf{k},t)$, and therefore (74) is equivalent to

$$v_0(\mathbf{k},t) = v_0^*(\mathbf{k}) - \frac{i}{k^3}\mathbf{k}^T\mathbf{q}_0(\mathbf{k},t), \tag{77}$$

and we therefore get from (71) that

$$W(k,t) = -k^3 v_0^*(\mathbf{k}), \tag{78}$$

for $t \geq 1$, which shows that the equation $W = 0$ is equivalent to choosing the initial condition

$$v_0^* = 0. \tag{79}$$

To overcome the problem related to the singular behavior of various expressions in (72)–(76) at $k = 0$, we now proceed exactly as in [10]. Namely, we substitute the integral equations (72)–(76) into the change of coordinates (62)–(65), and integrate by parts the time derivatives acting on $\mathbf{q}_0$. This leads to the following integral equations for τ, u, $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\begin{aligned}\tau(\mathbf{k},t) = &\left(\tilde{\tau}_-^*(\mathbf{k}) + \frac{1}{\Lambda_0}\mathbf{q}_0(\mathbf{k},1)\right)e^{\Lambda_-(t-1)}\\ &+ \frac{i\mathbf{k}^\perp}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)}q_1(\mathbf{k},s)\ ds + \frac{i\mathbf{k}^\perp}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)}q_1(\mathbf{k},s)\ ds\\ &- \frac{\Lambda_-}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)}\mathbf{q}_0(\mathbf{k},s)\ ds - \frac{\Lambda_+}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)}\mathbf{q}_0(\mathbf{k},s)\ ds,\end{aligned} \tag{80}$$

$$u(\mathbf{k},t) = -i\mathbf{k}^T\frac{1}{\Lambda_-}\left(\tilde{\tau}_-^*(\mathbf{k}) + \frac{1}{\Lambda_0}\mathbf{q}_0(\mathbf{k},1)\right)^\perp e^{\Lambda_-(t-1)} \tag{81}$$

$$
\begin{aligned}
&+ \left(\tilde{v}_-^*(\mathbf{k}) - \frac{i}{2k^2}\mathbf{k}^T \mathbf{q}_0(\mathbf{k},1)^\perp\right) e^{-k(t-1)} \\
&+ \frac{\Lambda_+}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)} q_1(\mathbf{k},s)\ ds - \frac{1}{2}\int_1^t e^{-k(t-s)} q_1(\mathbf{k},s)\ ds \\
&+ \frac{1}{2}\int_t^\infty e^{k(t-s)} q_1(\mathbf{k},s)\ ds + \frac{\Lambda_-}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)} q_1(\mathbf{k},s)\ ds \\
&- \frac{i}{2k}\mathbf{k}^T \int_1^t e^{-k(t-s)} \mathbf{q}_0(\mathbf{k},s)^\perp\ ds - \frac{i}{2k}\mathbf{k}^T \int_t^\infty e^{k(t-s)} \mathbf{q}_0(\mathbf{k},s)^\perp\ ds \\
&+ \frac{i}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)} \mathbf{k}^T \mathbf{q}_0(\mathbf{k},s)^\perp\ ds + \frac{i}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)} \mathbf{k}^T \mathbf{q}_0(\mathbf{k},s)^\perp\ ds,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}_1(\mathbf{k},t) = {}& -P_1\tau(\mathbf{k},t)^\perp + \frac{i}{k}\mathbf{k}\left(\tilde{v}_-^*(\mathbf{k}) - \frac{i}{2k^2}\mathbf{k}^T\mathbf{q}_0(\mathbf{k},1)^\perp\right) e^{-k(t-1)} \\
&- \frac{1}{2}\frac{i}{k}\mathbf{k}\int_1^t e^{-k(t-s)} q_1(\mathbf{k},s)\ ds - \frac{1}{2}\frac{i}{k}\mathbf{k}\int_t^\infty e^{k(t-s)} q_1(\mathbf{k},s)\ ds \\
&+ \frac{1}{2}\int_1^t e^{-k(t-s)} P_1\mathbf{q}_0(\mathbf{k},s)^\perp\ ds - \frac{1}{2}\int_t^\infty e^{k(t-s)} P_1\mathbf{q}_0(\mathbf{k},s)^\perp\ ds,
\end{aligned} \tag{82}
$$

$$
\begin{aligned}
\mathbf{v}_2(\mathbf{k},t) = {}& -\frac{1}{\Lambda_-}P_2\left(\tilde{\tau}_-^*(\mathbf{k}) + \frac{1}{\Lambda_0}\mathbf{q}_0(\mathbf{k},1)\right)^\perp e^{\Lambda_-(t-1)} \\
&+ \frac{1}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)} P_2\mathbf{q}_0(\mathbf{k},s)^\perp\ ds + \frac{1}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)} P_2\mathbf{q}_0(\mathbf{k},s)^\perp\ ds,
\end{aligned} \tag{83}
$$

with ω given by (69) and with $\mathbf{q}_0$ and q_1 given by (47) and (48), respectively. Note that the function η does not need to be constructed since it does not appear in the nonlinearities $\mathbf{q}_0$ and q_1.

4.1. Choice of initial conditions

A closer look at (80)–(83) reveals that the problem concerning the division by k in the equations (72)–(76) has not disappeared. However, in this new representation, the invariance properties of the equations have become manifest, and we see that the problem can be eliminated by a proper choice of initial conditions, *i.e.*, τ, u and $\mathbf{v}$ are either regular or singular for all times. In particular, as we will see, if we set

$$
\tilde{\tau}_-^*(\mathbf{k}) = \tau_-^*(\mathbf{k}) - \frac{1}{\Lambda_0}\mathbf{q}_0(\mathbf{k},1), \tag{84}
$$

$$
\tilde{v}_-^*(\mathbf{k}) = v_-^*(\mathbf{k}) + \frac{1}{2}\frac{i}{k^2}\mathbf{k}^T\mathbf{q}_0(\mathbf{k},1)^\perp, \tag{85}
$$

with

$$\tau_-^*(\mathbf{k}) = -i\mathbf{k}^\perp\ \tau_{-,1}^*(\mathbf{k}) - \Lambda_-\ P_1\tau_{-,2}^*(\mathbf{k}), \tag{86}$$

$$v_-^*(\mathbf{k}) = v_{-,1}^*(\mathbf{k}) - \frac{i\mathbf{k}^T}{k}\mathbf{v}_{-,2}^*(\mathbf{k})^\perp, \tag{87}$$

with $\tau_{-,1}^*$, $\tau_{-,2}^*$, $v_{-,1}^*$ and $\mathbf{v}_{-,2}^*$ smooth, then ω, τ, $\mathbf{q}_0$ and q_1 are smooth, and u, $\mathbf{v}_1$ and $\mathbf{v}_2$ are smooth modulo certain explicit discontinuities at $\mathbf{k} = 0$. This corresponds to choosing initial conditions exactly as singular as dictated by the nonlinearity. We expect this choice to be general enough to cover all cases of stationary exterior flows, but we do not address this question here.

Our choice of τ_-^* in (86) implies that $\mathbf{k}^T\tau_-^*(\mathbf{k})/k^2 = \mathbf{k}^T\tau_{-,2}^*(\mathbf{k})/\Lambda_+$, and therefore $\lim_{\mathbf{k}\to 0}\mathbf{k}^T\tau_-^*(\mathbf{k})/k^2 = 0$. This will translate below into $\omega(0) = 0$, which means that the longitudinal vorticity when averaged over transversal planes equals zero. This is dictated by the fact that, due to the divergence-freeness of the vorticity, this average is independent of the choice of the transversal plane, and should therefore be chosen to be equal to zero in our case (since there should be no nonzero vorticity average far ahead of an obstacle; see the introduction and [14] and [15] for the physical interpretation of the problem).

Below, we will prove existence of solutions to (80)–(83) for certain classes of continuous complex valued functions $\tau_{-,1}^*$, $v_{-,1}^*$ and continuous maps $\tau_{-,2}^*$, $\mathbf{v}_{-,2}^*$ with values in $\mathbf{C}^2$. Once the existence of solutions has been established, we will restrict attention to maps $\tau_{-,1}^*$, $\tau_{-,2}^*$, $v_{-,1}^*$ and $\mathbf{v}_{-,2}^*$ which satisfy, for all $\mathbf{k}\in\mathbf{R}^2$, $\overline{\tau_{-,1}^*(\mathbf{k})} = \tau_{-,1}^*(-\mathbf{k})$, $\overline{\tau_{-,2}^*(\mathbf{k})} = \tau_{-,2}^*(-\mathbf{k})$, $\overline{v_{-,1}^*(\mathbf{k})} = v_{-,1}^*(-\mathbf{k})$, and $\overline{\mathbf{v}_{-,2}^*(\mathbf{k})} = \mathbf{v}_{-,2}^*(-\mathbf{k})$ where the bar denotes the complex conjugate. This corresponds to the restriction to real valued solutions of (37)–(40).

Note that the parametrization (86) and (87) is not unique, *i.e.*, several choices of $\tau_{-,1}^*$, $\tau_{-,2}^*$, $v_{-,1}^*$ and $\mathbf{v}_{-,2}^*$ lead to the same τ_-^* and v_-^*. The parametrization has been chosen as given for convenience later on.

It turns out that, in contrast to the two-dimensional case, the decomposition of the nonlinearity $\mathbf{q}$ into $\mathbf{q}_0$ and q_1 is detailed enough to prove the existence of a solution of (80)–(83). This is due to the fact that, because of the dimension-dependence of power counting, the same nonlinearity has a smaller amplitude in three dimensions than in two dimensions (see for example [13] for the basics).

5. Function spaces

In order to prove the existence of a solution for (80)–(83) we will apply, for fixed initial conditions $r^* = (\tau_{-,1}^*, \tau_{-,2}^*, v_{-,1}^*\mathbf{v}_{-,2}^*)$, the contraction mapping principle to the map $(\tilde{\mathbf{q}}_0, \tilde{q}_1) = \mathcal{N}(\mathbf{q}_0, q_1)$ that is formally defined by first computing τ, u, $\mathbf{v}_1$ and $\mathbf{v}_2$ using (80)–(83), then ω and $\mathbf{v}$ using (69) and (65) and then $\mathbf{q}_0$ and q_1 by using (47) and (48).

To motivate our choice of functions spaces we recall that (see the introduction) the vorticity τ is expected to be a rapidly (faster than exponential) decaying function of $\mathbf{y}$ for all x. As a consequence, in Fourier space, τ and $\mathbf{q}$ ought to be smooth functions of $\mathbf{k}$ (probably entire), but to stay as general as possible we will assume as little smoothness as necessary, and it turns out to be sufficient to assume continuity of these functions. The decay properties of u and $\mathbf{v}$ in direct space are much less obvious, and we should therefore avoid to assume any smoothness in $\mathbf{k}$ that goes beyond what is necessary to show that $\lim_{\mathbf{y}\to\infty} u(x,\mathbf{y}) = \lim_{\mathbf{y}\to\infty} \mathbf{v}(x,\mathbf{y}) = 0$ in order to satisfy the boundary conditions. In Fourier space we will find these functions to be continuous, except for certain explicit singularities at the origin. The regularity in direct space of u, $\mathbf{v}$ and τ as a function of $\mathbf{y}$ translates into decay in Fourier space. This decay is parametrized below by the parameter α, which can essentially be chosen arbitrary. This is compatible with the fact that for small Reynolds numbers and exterior domains with smooth boundaries solutions of the Navier–Stokes equations are known to be arbitrary smooth in direct space (see for example [7]). More details are given in Section 9.

With these ideas in mind we now define the functions spaces that will be used below: Let, for α, $p \geq 0$ and $k \geq 0$,

$$\mu_\alpha^p(k,t) = \frac{1}{1+(kt^p)^\alpha}. \tag{88}$$

Let furthermore

$$\mu_\alpha(k,t) = \mu_\alpha^{1/2}(k,t),$$
$$\bar{\mu}_\alpha(k,t) = \mu_\alpha^1(k,t).$$

We then consider, for fixed $\alpha \geq 0$ and $\nu \in \mathbf{N}$, the Banach spaces $\mathcal{V}_\alpha^\nu$ of continuous complex valued maps $\mathbf{f} \equiv (f_1, \ldots, f_\nu) \in \mathcal{C}(\mathbf{R}^2, \mathbf{C}^\nu)$ equipped with the norm

$$\|\mathbf{f}\|_\alpha = \sup_{\mathbf{k}\in\mathbf{R}^2} \frac{|\mathbf{f}(\mathbf{k})|}{\mu_\alpha(|\mathbf{k}|,1)},$$

where,

$$|\mathbf{f}(\mathbf{k})| = \sum_{i=1,\ldots,\nu} |f_i(\mathbf{k})|,$$

and the Banach space $\mathcal{B}_{\alpha,\beta}^\nu$ of continuous maps $\mathbf{f}$ from $[1,\infty)$ to $\mathcal{V}_\alpha^\nu$ equipped the norm

$$\|\mathbf{f}\|_{\alpha,\beta} = \sup_{t\geq 1} t^\beta ||\mathbf{f}(t^{-1/2}.,t)||_\alpha.$$

Finally, we define the Banach space

$$\mathcal{V}_\alpha = \mathcal{V}_\alpha^1 \oplus \mathcal{V}_\alpha^2 \oplus \mathcal{V}_\alpha^1 \oplus \mathcal{V}_\alpha^2,$$

equipped with the norm

$$\|(f_1,f_2,f_3,f_4)\|_\alpha = \sum_{i=1,\ldots,4} \|f_i\|_\alpha,$$

and the Banach space $\mathcal{B}_\alpha$,

$$\mathcal{B}_\alpha = \mathcal{B}^2_{\alpha,3/2} \oplus \mathcal{B}^1_{\alpha,3/2},$$

equipped with the norm

$$\|(f_1, f_2)\|_\alpha = \|f_1\|_{\alpha,3/2} + \|f_2\|_{\alpha,3/2} .$$

Theorem 2. *Let $\alpha > 2$. Let $r^* = (\tau^*_{-,1}, \tau^*_{-,2}, v^*_{-,1}, \mathbf{v}^*_{-,2}) \in \mathcal{V}_{\alpha+1}$ with $\varepsilon_0 = \|r^*\|_{\alpha+1}$. Then, $\mathcal{N}$ is well defined as a map from $\mathcal{B}_\alpha$ to $\mathcal{B}_\alpha$ and contracts, for ε_0 sufficiently small, the ball $B_\alpha(\varepsilon_0) = \{\rho \in \mathcal{B}_\alpha \mid \|\rho\|_\alpha \le \varepsilon_0\}$ into itself.*

Theorem 2 implies that for ε_0 small enough $\mathcal{N}$ has a unique fixed point in $B_\alpha(\varepsilon_0)$, *i.e.*, the integral equations (80)–(83) have a solution.

6. Proof of Theorem 2

The proof is organized as follows: we first show that $\mathcal{N}$ is well defined and maps, for small enough initial conditions $r^* = (\tau^*_{-,1}, \tau^*_{-,2}, v^*_{-,1}, \mathbf{v}^*_{-,2})$, a ball in $\mathcal{B}_\alpha$ into itself. Then, we show that $\mathcal{N}$ is a contraction on this ball.

Let ε_0 be as in Theorem 2. Throughout all proofs we then denote by ε a constant multiple of ε_0, *i.e.*, $\varepsilon = \text{const.}\ \varepsilon_0$ with a constant that may be different from instance to instance. Also, to simplify notation, we will write throughout all proofs k instead of $|\mathbf{k}|$.

6.1. $\mathcal{N}$ is well defined

We first prove bounds on τ, u, $\mathbf{v}_1$ and $\mathbf{v}_2$:

Proposition 3. *Let $\alpha > 0$. Let $r^* = (\tau^*_{-,1}, \tau^*_{-,2}, v^*_{-,1}, \mathbf{v}^*_{-,2}) \in \mathcal{V}_{\alpha+1}$ with $\varepsilon_0 = \|\mathbf{r}^*\|_{\alpha+1}$, and let $(\mathbf{q}_0, q_1) \in B_\alpha(\varepsilon)$. Then, ω and τ as defined by* (69) *and* (80) *are continuous maps from $\mathbf{R}^2 \times [1,\infty)$ to $\mathbf{C}^2$ and $\mathbf{C}$, respectively, and u, $\mathbf{v}_1$ and $\mathbf{v}_2$ as defined by* (81), (82) *and* (83) *are of the form*

$$u(\mathbf{k},t) = u_E(\mathbf{k},t) + \frac{1}{k} i\mathbf{k}^T \mathbf{u}_O(\mathbf{k},t), \tag{89}$$

$$\mathbf{v}_1(\mathbf{k},t) = \mathbf{v}_{1,C}(\mathbf{k},t) + P_1 \mathbf{v}_{1,E}(\mathbf{k},t) + \frac{i\mathbf{k}^T}{k} v_{1,O}(\mathbf{k},t), \tag{90}$$

$$\mathbf{v}_2(\mathbf{k},t) = P_2 \mathbf{v}_{2,E}(\mathbf{k},t), \tag{91}$$

with u_E, $\mathbf{u}_O$, $\mathbf{v}_{1,C}$, $\mathbf{v}_{1,E}$, $v_{1,O}$ and $\mathbf{v}_{2,E}$ continuous maps from $\mathbf{R}^2 \times [1,\infty)$ to $\mathbf{C}$ and $\mathbf{C}^2$, respectively. Furthermore, we have the bounds

$$|\omega(\mathbf{k},t)| \le \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t), \tag{92}$$

$$|\tau(\mathbf{k},t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t), \tag{93}$$

$$|u(\mathbf{k},t)| \leq \varepsilon\mu_\alpha(k,t), \tag{94}$$

$$|\mathbf{v}_1(\mathbf{k},t)| \leq \varepsilon\bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t), \tag{95}$$

$$|\mathbf{v}_2(\mathbf{k},t)| \leq \varepsilon\mu_{\alpha+1}(k,t), \tag{96}$$

uniformly in $\mathbf{k} \in \mathbf{R}^2$ *and* $t \geq 1$.

See Appendix II for a proof.

Now we prove bounds on $\mathbf{q}_0$ and q_1:

Proposition 4. *Let* $\alpha > 2$. *Let* ω *and* τ *be continuous maps from* $\mathbf{R}^2 \times [1,\infty)$ *to* $\mathbf{C}$ *and* $\mathbf{C}^2$ *satisfying the bounds* (92) *and* (93), *respectively, and let* u, $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *by continuous maps from* $\mathbf{R}^2 \setminus \{0\} \times [1,\infty)$ *to* $\mathbf{C}$ *and* $\mathbf{C}^2$, *respectively, satisfying the bounds* (94)–(96). *Then,* $\mathbf{q}_0$ *and* q_1 *as defined by* (47) *and* (48) *are continuous maps from* $\mathbf{R}^2 \times [0,\infty)$ *to* $\mathbf{C}^2$ *and* $\mathbf{C}$, *respectively, and we have the bounds*

$$|\mathbf{q}_0(\mathbf{k},t)| \leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t), \tag{97}$$

$$|q_1(\mathbf{k},t)| \leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t), \tag{98}$$

uniformly in $\mathbf{k} \in \mathbf{R}^2$ *and* $t \geq 1$, *and therefore* $\|(\mathbf{q}_0, q_1)\|_\alpha \leq \varepsilon^2$.

Proof. The bounds (95) and (96) imply that

$$|\mathbf{v}(\mathbf{k},t)| \leq \varepsilon\mu_\alpha(k,t). \tag{99}$$

We first prove the bound on $\mathbf{q}_0$. Namely, using Proposition 16 (see Appendix V), we find from (92, (93), (94) and (99) that $|(u * \tau)(\mathbf{k},t)| \leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t)$ and that $|(\omega * \mathbf{v})(\mathbf{k},t)| \leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t)$ and (97) follows using the triangle inequality. Similarly we have for q_1, using (99) and (93) that $\left|\left(\mathbf{v}^T * \tau^\perp\right)(\mathbf{k},t)\right| \leq \frac{\varepsilon^2}{t^{3/2}}\mu_\alpha(k,t)$, which proves (98). □

Proposition 3 together with Proposition 4 imply that, for $\rho \in B_\alpha(\varepsilon)$, $\|\mathcal{N}(\rho)\|_\alpha \leq \varepsilon^2$. Therefore, $\mathcal{N}$ is well defined as a map from $\mathcal{B}_\alpha$ to $\mathcal{B}_\alpha$. Furthermore, since $\varepsilon^2 = \text{const.}\ \varepsilon_0^2$, it follows that $\mathcal{N}$ maps $B_\alpha(\varepsilon_0)$ into itself for ε_0 small enough.

6.2. $\mathcal{N}$ is Lipschitz

In order to complete the proof of Theorem 2 it remains to be shown that $\mathcal{N}$ is Lipschitz:

Proposition 5. *Let* $\alpha > 2$. *Let* $r^* = (\tau^*_{-,1}, \tau^*_{-,2}, v^*_{-,1}, \mathbf{v}^*_{-,2}) \in \mathcal{V}_{\alpha+1}$ *with* $\varepsilon_0 = \|\mathbf{r}^*\|_{\alpha+1}$, *and let* ρ, $\tilde{\rho} \in B_\alpha(\varepsilon_0)$. *Then*

$$\|\mathcal{N}(\rho) - \mathcal{N}(\tilde{\rho})\|_\alpha \leq \varepsilon \|\rho - \tilde{\rho}\|_\alpha . \tag{100}$$

Proof. Let $\rho^1 \equiv (\rho^1_1, \rho^1_2)$, $\rho^2 \equiv (\rho^2_1, \rho^2_2) \in B_\alpha(\varepsilon_0)$. Then, by Proposition 3 and Proposition 4, $\rho \equiv \mathcal{N}(\rho^1) - \mathcal{N}(\rho^2)$ is well defined and $\rho \in \mathcal{B}_\alpha$. Let $\rho \equiv (\rho_1, \rho_2)$, and let ω^i, τ^i, u^i, $\mathbf{v}^i$, $i = 1, 2$, be the quantities (69), (80), (81) and (65) computed from ρ^1 and ρ^2, respectively. Using the identity $ab - \tilde{a}\tilde{b} = (a - \tilde{a})b + \tilde{a}(b - \tilde{b})$ (distributive law) we find that

$$\begin{aligned}
\rho_1 &= \frac{1}{4\pi^2}\left(u^1 * \tau^1 - \omega^1 * \mathbf{v}^1\right) - \frac{1}{4\pi^2}\left(u^2 * \tau^2 - \omega^2 * \mathbf{v}^2\right) \\
&= \frac{1}{4\pi^2}\left[\left(u^1 - u^2\right) * \tau^1 + u^2 * \left(\tau^1 - \tau^2\right)\right] \\
&\quad - \frac{1}{4\pi^2}\left[\left(\omega^1 - \omega^2\right) * \mathbf{v}^1 + \omega^2 * \left(\mathbf{v}^1 - \mathbf{v}^2\right)\right],
\end{aligned}$$

and similarly that

$$\rho_2 = -\frac{1}{4\pi^2}\left[\left(\mathbf{v}^1 - \mathbf{v}^2\right)^T * (\tau^1)^\perp + \left(\mathbf{v}^2\right)^T * (\tau^1 - \tau^2)^\perp\right].$$

Therefore, and since the quantities ω^i, τ^i, u^i, $\mathbf{v}^i$, $i = 1, 2$ are linear (respectively affine) in ρ^1 and ρ^2, the bound (100) follows *mutatis mutandis* from the proofs of Proposition 3 and Proposition 4. □

Proposition 3 together with Proposition 4 show that, for $\alpha > 2$, $\mathcal{N}$ maps the ball $B_\alpha(\varepsilon_0)$ into itself for ε_0 small enough, and Proposition 5 therefore shows that $\mathcal{N}$ is a contraction of $B_\alpha(\varepsilon_0)$ into itself for ε_0 small enough. This completes the proof of Theorem 2.

7. Invariant quantities

We again restrict attention to maps $\tau^*_{-,1}$, $\tau^*_{-,2}$, $v^*_{-,1}$ and $\mathbf{v}^*_{-,2}$ which correspond to real valued solutions of (37)–(40).

Proposition 6. *Let* $\mathbf{k} = k\mathbf{e}$ *with* $\mathbf{e}$ *a unit vector, let*

$$\mathbf{a} = \left(\tau^*_{-,2}(0) + \int_1^\infty \mathbf{q}_0(0, s)\ ds\right)^\perp, \tag{101}$$

$$\mathbf{b} = -\left(\mathbf{v}^*_{-,2}(0) + \frac{1}{2}\int_1^\infty \mathbf{q}_0(0, s)\ ds\right)^\perp, \tag{102}$$

$$c = -\tau^*_{-,1}(0) + \int_1^\infty q_1(0, s)\ ds, \tag{103}$$

$$d = v^*_{-,1}(0) - \frac{1}{2} \int_1^\infty q_1(0,s)\ ds, \tag{104}$$

and let

$$\mathbf{r}(t) = -\left(\int_t^\infty (1 - e^{t-s}) \mathbf{q}_0(0,s)\ ds \right)^\perp . \tag{105}$$

Then, in the limit $k \to 0$, the equations (81), (82) *and* (83) *reduce to*

$$\lim_{k\to 0} u(k\mathbf{e}, t) = c + d + i\mathbf{e}^T \mathbf{b}, \tag{106}$$

$$\lim_{k\to 0} \mathbf{v}_1(k\mathbf{e}, t) = -P_1\mathbf{b} + P_1\mathbf{r}(t) + i\mathbf{e}d, \tag{107}$$

$$\lim_{k\to 0} \mathbf{v}_2(k\mathbf{e}, t) = P_2\mathbf{a} + P_2\mathbf{r}(t). \tag{108}$$

Proof. This follows immediately using Proposition 3. □

From (106)–(108) we can extract the time independent (real) quantities $\mathbf{a}$, $\mathbf{b}$, c and d. Namely, let $\mathbf{e} \equiv \mathbf{e}(\vartheta) = (\cos(\vartheta), \sin(\vartheta))$, and let the average $\langle\ .\ \rangle$ be as defined in (15). Then, we see from (106), (107) and (108) that

$$\left\langle \lim_{k\to 0} \mathbf{v}_1(k\mathbf{e}, t) \right\rangle = -\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{r}(t),$$

$$\left\langle \lim_{k\to 0} \mathbf{v}_2(k\mathbf{e}, t) \right\rangle = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{r}(t),$$

and therefore we have, for any $t \geq 1$,

$$d = \left\langle -i\mathbf{e}^T \lim_{k\to 0} \mathbf{v}_1(k\mathbf{e}, t) \right\rangle,$$

$$c + d = \left\langle \lim_{k\to 0} u(k\mathbf{e}, t) \right\rangle,$$

$$\mathbf{b} = \left\langle -i\mathbf{e} \lim_{k\to 0} u(k\mathbf{e}, t) \right\rangle,$$

$$\mathbf{a} + \mathbf{b} = \left\langle 2 \lim_{k\to 0} \mathbf{v}_2(k\mathbf{e}, t) \right\rangle - \left\langle 2 \lim_{k\to 0} \mathbf{v}_1(k\mathbf{e}, t) \right\rangle.$$

Equations (101)–(104) imply furthermore that the quantities ϕ and ψ,

$$\phi = c + 2d = -\tau^*_{-,1}(0) + 2v^*_{-,1}(0), \tag{109}$$

$$\psi = \mathbf{a} + 2\mathbf{b} = \left(\tau^*_{-,2}(0) - 2\mathbf{v}^*_{-,2}(0)\right)^\perp, \tag{110}$$

are not only invariant but are directly specified by the choice of the initial conditions, *i.e.*, their value is known without solving the equations. In particular, as indicated in the introduction, ϕ and ψ should be chosen equal to zero for the case of the downstream asymptotics of a stationary flow around a body. More precisely we have that $\phi = 0$ (zero flux at infinity; see [1]) and that $\psi = 0$ (matching of asymptotic terms at $x = 0$; see Appendix VI for more details). We also expect the constants $c = -2d$ and $\mathbf{a} = -2\mathbf{b}$ to be related to the drag and lift exerted on the body. For a similar interpretation in the two-dimensional case see [14], [15] and [2], [3].

8. Asymptotic behavior

The following theorem provides the leading order behavior of solutions whose existence has been shown in Theorem 2. We again restrict attention to maps $\tau^*_{-,1}$, $\tau^*_{-,2}$, $v^*_{-,1}$ and $\mathbf{v}^*_{-,2}$ which correspond to real valued solutions of (37)–(40).

Theorem 7. *Let* $\alpha > 2$. *Let* $r^* = (\tau^*_{-,1}, \tau^*_{-,2}, v^*_{-,1}, \mathbf{v}^*_{-,2}) \in \mathcal{V}_{\alpha+1}$ *with* $\varepsilon_0 = \|\mathbf{r}^*\|_{\alpha+1}$ *sufficiently small. Then, the equations* (80)–(83) *have a solution and*

$$\lim_{t\to\infty} t \int_{\mathbf{R}} |u(\mathbf{k},t) - u_{as}(\mathbf{k},t)| \ d^2\mathbf{k} \ = 0, \tag{111}$$

$$\lim_{t\to\infty} t^{3/2} \int_{\mathbf{R}} |\mathbf{v}_1(\mathbf{k},t) - \mathbf{v}_{1,as}(\mathbf{k},t)| \ d^2\mathbf{k} \ = 0, \tag{112}$$

$$\lim_{t\to\infty} t \int_{\mathbf{R}} |\mathbf{v}_2(\mathbf{k},t) - \mathbf{v}_{2,as}(\mathbf{k},t)| \ d^2\mathbf{k} \ = 0, \tag{113}$$

where

$$u_{as}(\mathbf{k},t) = e^{-k^2t} \ c + e^{-kt} \ d + \frac{i\mathbf{k}^T}{k} e^{-kt} \ \mathbf{b}, \tag{114}$$

$$\mathbf{v}_{1,as}(\mathbf{k},t) = i\mathbf{k}e^{-k^2t} \ c + \frac{i}{k}\mathbf{k}e^{-kt} \ d - P_1 e^{-kt} \ \mathbf{b}, \tag{115}$$

$$\mathbf{v}_{2,as}(\mathbf{k},t) = P_2 e^{-k^2t} \ \mathbf{a} = e^{-k^2t} \ \mathbf{a} - P_1 e^{-k^2t} \ \mathbf{a} \tag{116}$$

with $\mathbf{a}$, $\mathbf{b}$, c *and* d *as defined in* (101)–(104).

The existence of a solution follows from Theorem 2. A proof of (111), (112) and (113) can be found in Appendix III.

9. Proof of Theorem 1

We again restrict attention to maps $\tau^*_{-,1}$, $\tau^*_{-,2}$, $v^*_{-,1}$ and $\mathbf{v}^*_{-,2}$ which correspond to real valued solutions of (37)–(40). For $\alpha > 2$ we have proved in Section 6 the existence of a solution of the equations (80)–(83) or respectively (45), satisfying (to avoid confusion we now write the hats for the Fourier transforms)

$$|\hat{u}(\mathbf{k},t)| \leq \varepsilon\mu_\alpha(k,t), \tag{117}$$

$$|\hat{\mathbf{v}}_1(\mathbf{k},t)| \leq \varepsilon\bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t), \tag{118}$$

$$|\hat{\mathbf{v}}_2(\mathbf{k},t)| \leq \varepsilon\mu_{\alpha+1}(k,t). \tag{119}$$

Since, for $\alpha > 2$, the real and imaginary parts of the functions $\mathbf{k} \mapsto \hat{u}(\mathbf{k},t)$, $\mathbf{k} \mapsto \hat{\mathbf{v}}_1(\mathbf{k},t)$ and $\mathbf{k} \mapsto \hat{\mathbf{v}}_2(\mathbf{k},t)$ are, respectively, even and odd functions in $L^1(\mathbf{R}^2, d^2\mathbf{k})$ for all $t \geq 1$, their Fourier transforms

$$u(x,\mathbf{y}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbf{R}^2} e^{-i\mathbf{k}\cdot\mathbf{y}}\ \hat{u}(\mathbf{k},x)\ d^2\mathbf{k},$$

$$\mathbf{v}_1(x,\mathbf{y}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbf{R}^2} e^{-i\mathbf{k}\cdot\mathbf{y}}\ \hat{\mathbf{v}}_1(\mathbf{k},x)\ d^2\mathbf{k},$$

$$\mathbf{v}_2(x,\mathbf{y}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbf{R}^2} e^{-i\mathbf{k}\cdot\mathbf{y}}\ \hat{\mathbf{v}}_2(\mathbf{k},x)\ d^2\mathbf{k},$$

are by the Riemann–Lebesgue lemma real valued continuous maps of $\mathbf{y}$ and vanish as $|\mathbf{y}| \to \infty$ for each $x \geq 1$. Moreover, using (117), (118) and (119), we find that, for $x \geq 1$,

$$\sup_{\mathbf{y}\in\mathbf{R}^2} |u(x,\mathbf{y})| \leq \frac{\varepsilon}{|x|}, \tag{120}$$

$$\sup_{\mathbf{y}\in\mathbf{R}^2} |\mathbf{v}_1(x,\mathbf{y})| \leq \frac{\varepsilon}{|x|^{3/2}}, \tag{121}$$

$$\sup_{\mathbf{y}\in\mathbf{R}^2} |\mathbf{v}_2(x,\mathbf{y})| \leq \frac{\varepsilon}{|x|}. \tag{122}$$

As a consequence, u and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ converge to zero whenever $|x| + |\mathbf{y}| \to \infty$ in Ω (see Section 5 of [14] for a detailed proof), and $\mathbf{u} = \mathbf{u}_\infty + (u, \mathbf{v})$ satisfies therefore not only (25) but also the boundary conditions (3), (4). The reconstruction of the pressure from u and $\mathbf{v}$ is standard. For $\alpha > 4$ second derivatives of u and $\mathbf{v}$ are continuous in direct space, and one easily verifies using the definitions that the pair $(\mathbf{u}, p)$ satisfies the Navier–Stokes equations (1), (2). The set $\mathcal{S}$ in Theorem 1 is by definition the set of all vector fields $(u, \mathbf{v})$ obtained this way, restricted to Σ. Finally, equations (5)–(7) are a direct consequence of Theorem 7 (see Appendix VI for the computation of the Fourier transforms). This completes the proof of Theorem 1.

10. Appendix I

In this appendix we construct a matrix S, with the same block structure as L,

$$S = \begin{pmatrix} S_1 & 0 \\ S_3 & S_2 \end{pmatrix}, \tag{123}$$

such that

$$S^{-1}LS = D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \tag{124}$$

with D_1 a diagonal 4×4 matrix with diagonal entries Λ_+, Λ_+, Λ_-, Λ_- and with D_2 diagonal 3×3 matrix with diagonal entries 0, k, and $-k$. (Note the branch of zero modes which is not present in the two-dimensional case.) The matrix S_1 diagonalizes L_1. Namely, $D_1 = S_1^{-1}L_1S_1$, where S_1 is given in (56), the entries being 2×2 matrices. The inverse of S_1 is given in (59), the entries again being 2×2 matrices. The matrix S_2 diagonalizes L_2, namely $D_2 = S_2^{-1}L_2S_2$, where S_2 is given in (56), the first line being 1×1 matrices and the second line 2×1 matrices. The inverse of S_2 is given in (59), the first column being 1×1 matrices and the second column 1×2 matrices.

We now compute S_3. Since S has to satisfy $LS = SD$, we find for S_3 the equation $L_3S_1 + L_2S_3 = S_3D_1$, which can be solved as follows. Let $S_3 = S_2Y$. Then, using that $L_2 = S_2D_2S_2^{-1}$, we get the following equation for the matrix Y,

$$S_2^{-1}L_3S_1 = -D_2Y + YD_1,$$

which can be solved for Y entry by entry, *i.e.*,

$$Y_{ij} = \frac{1}{-\left(D_2\right)_{ii} + \left(D_1\right)_{jj}} \left(S_2^{-1}L_3S_1\right)_{ij},$$

for $i = 1, \ldots, 3$, $j = 1, \ldots, 4$. Explicitly, we have the 3×4 matrix

$$L_3S_1(\mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix},$$

and therefore

$$S_2^{-1}L_3S_1(\mathbf{k}) = \begin{pmatrix} \frac{i}{k}\mathbf{k}^T & \frac{i}{k}\mathbf{k}^T \\ \frac{i}{2k}\left(\mathbf{k}^\perp\right)^T & \frac{i}{2k}\left(\mathbf{k}^\perp\right)^T \\ \frac{-i}{2k}\left(\mathbf{k}^\perp\right)^T & \frac{-i}{2k}\left(\mathbf{k}^\perp\right)^T \end{pmatrix},$$

which leads to

$$Y(\mathbf{k}) = \begin{pmatrix} \frac{1}{\Lambda_+}\frac{i}{k}\mathbf{k}^T & \frac{1}{\Lambda_-}\frac{i}{k}\mathbf{k}^T \\ \frac{1}{\Lambda_+-k}\frac{i}{2k}\left(\mathbf{k}^\perp\right)^T & \frac{1}{\Lambda_--k}\frac{i}{2k}\left(\mathbf{k}^\perp\right)^T \\ \frac{1}{\Lambda_++k}\frac{-i}{2k}\left(\mathbf{k}^\perp\right)^T & \frac{1}{\Lambda_-+k}\frac{-i}{2k}\left(\mathbf{k}^\perp\right)^T \end{pmatrix}.$$

Using moreover the identities

$$\frac{1}{\Lambda_+ - k} - \frac{1}{\Lambda_+ + k} = \frac{2k}{\Lambda_+}, \qquad \frac{1}{\Lambda_- - k} - \frac{1}{\Lambda_- + k} = \frac{2k}{\Lambda_-},$$

$$\frac{1}{\Lambda_+ - k} + \frac{1}{\Lambda_+ + k} = \frac{1}{\Lambda_- - k} + \frac{1}{\Lambda_- + k} = 2,$$

we finally get for S_3 the matrix (57). We also need S^{-1}. We find that

$$S^{-1} = \begin{pmatrix} S_1^{-1} & 0 \\ (S^{-1})_3 & S_2^{-1} \end{pmatrix},$$

with $(S^{-1})_3 = -S_2^{-1} S_3 S_1^{-1} = -Y S_1^{-1}$, for which we find (60).

11. Appendix II

In this appendix we prove Proposition 3. We first prove the continuity, then the bounds. Throughout this and subsequent sections we make extensive use of Proposition 15 (see Appendix V).

We note that the maps u, $\mathbf{v}_1$ and $\mathbf{v}_2$ as defined in (81), (82) and (83) are explicitly of the form indicated in (89)–(91). The continuity of the maps u_E, $\mathbf{u}_O$, $\mathbf{v}_{1,C}$, $\mathbf{v}_{1,E}$, $v_{1,O}$ and $\mathbf{v}_{2,E}$ is elementary, since all the integrals converge uniformly in k. Namely, we have that $|\mathbf{q}_0(\mathbf{k}, s)| \leq \varepsilon/s^{3/2}$ and $|q_1(\mathbf{k}, s)| \leq \varepsilon/s^{3/2}$ uniformly in $k \geq 0$, and that $1/s^{3/2}$ is integrable at infinity.

11.1. Bounds on $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$

The bound (92) follows immediately from (96) using the definition (69) of ω. Next, we write $\tau = \sum_{i=1}^{5} \tau_i$, with τ_i the i-th term in (80) and we bound each of the terms individually. The inequality (93) then follows using the triangle inequality.

Proposition 8. *For all $\alpha \geq 0$ and $i = 1, \ldots, 5$ we have the bounds*

$$|\tau_i(\mathbf{k}, t)| \leq \frac{\varepsilon}{t^{\sigma_i}} \mu_\alpha(k, t), \tag{125}$$

with $\sigma_1 = \sigma_2 = \frac{1}{2}$, $\sigma_4 = 1$ and $\sigma_3 = \sigma_5 = \frac{3}{2}$, uniformly in $t \geq 1$ and $\mathbf{k}^2 \in \mathbf{R}^2$.

First, for τ_1 we have, using that $k^2 = -\Lambda_- \Lambda_+$, that furthermore $\Lambda_+^{1/2} \mu_{\alpha+1}(k, 1) \leq \text{const.} \mu_{\alpha+1/2}(k, 1)$ and using Proposition 15 (see Appendix V) that

$$\begin{aligned}
&\left| \left(-i\mathbf{k}^{\perp}\ \tau^*_{-,1}(\mathbf{k}) - \Lambda_-\ P_1 \tau^*_{-,2}(\mathbf{k}) \right) e^{\Lambda_-(t-1)} \right| \\
&\leq \varepsilon \left(\mu_{\alpha+1/2}(k, 1)\, |\Lambda_-|^{1/2} + \mu_{\alpha+1}(k, 1)\, |\Lambda_-| \right) e^{\Lambda_-(t-1)} \\
&\leq \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k, t) + \frac{\varepsilon}{t} \mu_\alpha(k, t),
\end{aligned}$$

and (125) follows for $i = 1$. Next, splitting the integral in the definition of τ_2 into two parts we find that

$$\begin{aligned}\left|\frac{1}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}q_1(\mathbf{k},s)\ ds\right| &\leq \varepsilon\mu_{\alpha+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}\int_1^{\frac{t+1}{2}}\frac{1}{s^{3/2}}ds\\ &\leq \varepsilon\mu_{\alpha+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}\left(\frac{t-1}{t}\right)\\ &\leq \varepsilon\mu_{\alpha+1}(k,t),\end{aligned}$$

and that

$$\begin{aligned}\left|\frac{1}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}q_1(\mathbf{k},s)\ ds\right| &\leq \varepsilon\frac{1}{\Lambda_0}\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t}\frac{1}{s^{3/2}}ds\\ &\leq \frac{\varepsilon}{t^{1/2}}\frac{1}{\Lambda_0}\mu_\alpha(k,t)\leq\varepsilon\mu_{\alpha+1}(k,t).\end{aligned}$$

Therefore we get, using the triangle inequality, that

$$\left|\frac{1}{\Lambda_0}\int_1^{t} e^{\Lambda_-(t-s)}q_1(\mathbf{k},s)\ ds\right| \leq \varepsilon\mu_{\alpha+1}(k,t). \tag{126}$$

The bound (125) now follows for $i=2$, using that $\varepsilon k\mu_{\alpha+1}(k,t)\leq\varepsilon\mu_\alpha(k,t)/t^{1/2}$. For τ_3 we have that

$$\begin{aligned}\left|\frac{i\mathbf{k}^\perp}{\Lambda_0}\int_t^{\infty} e^{\Lambda_+(t-s)}q_1(\mathbf{k},s)\ ds\right| &\leq \frac{\varepsilon}{t^{3/2}}\frac{|k|}{\Lambda_0}\mu_\alpha(k,t)\int_t^{\infty}e^{\Lambda_+(t-s)}ds\\ &\leq \frac{\varepsilon}{t^{3/2}}\frac{1}{\Lambda_+}\frac{|k|}{\Lambda_0}\mu_\alpha(k,t)\leq\frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t),\end{aligned}$$

which proves (125) for $i=3$. The integral defining τ_4 we split into two parts. We have that

$$\begin{aligned}\left|\frac{\Lambda_-}{\Lambda_0}\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}\mathbf{q}_0(\mathbf{k},s)\ ds\right| &\leq \varepsilon\mu_{\alpha+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}\left|\Lambda_-\right|\int_1^{\frac{t+1}{2}}\frac{1}{s^{3/2}}ds\\ &\leq \varepsilon\mu_{\alpha+1}(k,1)e^{\Lambda_-\frac{t-1}{2}}\left|\Lambda_-\right|\left(\frac{t-1}{t}\right)\\ &\leq \frac{\varepsilon}{t}\mu_{\alpha+1}(k,t),\end{aligned}$$

and that

$$\left|\frac{\Lambda_-}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\mathbf{q}_0(\mathbf{k},s)\ ds\right| \leq \frac{\varepsilon}{t}\frac{1}{\Lambda_0}\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t}e^{\Lambda_-(t-s)}\left|\Lambda_-\right|ds\leq\frac{\varepsilon}{t}\frac{1}{\Lambda_0}\mu_\alpha(k,t),$$

and (125) follows for $i=4$ using the triangle inequality. For τ_5 we finally have that

$$\left|\frac{\Lambda_+}{\Lambda_0}\int_t^\infty e^{\Lambda_+(t-s)}\mathbf{q}_0(\mathbf{k},s)\ ds\right| \leq \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t)\int_t^\infty e^{\Lambda_+(t-s)}ds$$
$$\leq \frac{\varepsilon}{t^{3/2}}\frac{1}{\Lambda_+}\mu_\alpha(k,t) \leq \frac{\varepsilon}{t^{3/2}}\mu_\alpha(k,t),$$

which proves (125) for $i=5$.

11.2. Bound on u

We write $u=\sum_{i=1}^{10}u_i$, with u_i the i-th term in (81), and we bound each of the terms individually. The inequality (94) then follows using the triangle inequality.

Proposition 9. *For all $\alpha\geq 0$ and $i\in\{1,3,5,6,8,9,10\}$ we have the bound*

$$|u_i(\mathbf{k},t)| \leq \frac{\varepsilon}{t^{\sigma_i}}\mu_\alpha(k,t), \tag{127}$$

with $\sigma_1=\sigma_3=0$, $\sigma_5=\sigma_8=\sigma_9=\frac{1}{2}$ and $\sigma_6=\sigma_{10}=\frac{3}{2}$ and for $i\in\{2,4,7\}$ we have the bound

$$|u_i(\mathbf{k},t)| \leq \varepsilon\bar{\mu}_{\alpha+1}(k,t)+\sigma_i\frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t), \tag{128}$$

with $\sigma_2=0$ and $\sigma_4=\sigma_7=1$, uniformly in $t\geq 1$ and $\mathbf{k}\in\mathbf{R}^2$.

For u_1 we have

$$\left|-i\mathbf{k}^T\frac{1}{\Lambda_-}\left(-i\mathbf{k}^\perp\ \tau^*_{-,1}(\mathbf{k})\right)^\perp e^{\Lambda_-(t-1)}\right| \leq \left|\Lambda_+\tau^*_{-,1}(\mathbf{k})e^{\Lambda_-(t-1)}\right|$$
$$\leq \varepsilon\mu_\alpha(k,1)e^{\Lambda_-(t-1)},$$

and (127) follows for $i=1$ using Proposition 15. For u_2 we have,

$$\left|\left(v^*_{-,1}(\mathbf{k})-\frac{i\mathbf{k}^T}{k}\mathbf{v}^*_{-,2}(\mathbf{k})^\perp\right)e^{-k(t-1)}\right| \leq \mu_{\alpha+1}(k,1)e^{-k(t-1)}$$

and (128) follows for $i=2$ using Proposition 15. For u_3 we use (126) and get

$$\left|\frac{\Lambda_+}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)}q_1(\mathbf{k},s)\ ds\right| \leq \varepsilon\Lambda_+\mu_{\alpha+1}(k,t),$$

and (127) follows for $i=3$. The integral defining u_4 we split into two parts. We have that

$$\left|\frac{1}{2}\int_1^{\frac{t+1}{2}} e^{-k(t-s)}q_1(\mathbf{k},s)\ ds\right| \leq \varepsilon e^{-k\frac{t-1}{2}}\mu_\alpha(k,1)\int_1^{\frac{t+1}{2}}\frac{1}{s^{3/2}}\ ds$$
$$\leq \varepsilon\mu_\alpha(k,1)e^{-k\frac{t-1}{2}}\left(\frac{t-1}{t}\right) \leq \varepsilon\bar{\mu}_{\alpha+1}(k,t),$$

and that

$$\left|\frac{1}{2}\int_{\frac{t+1}{2}}^{t} e^{-k(t-s)} q_1(\mathbf{k},s)\ ds\right| \leq \varepsilon\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t} \frac{1}{s^{3/2}}\ ds \leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t),$$

and (128) follows for $i=4$. Next, to bound u_5, we use that

$$\left|\frac{1}{2}\int_{t}^{\infty} e^{k(t-s)} q_1(\mathbf{k},s)\ ds\right| \leq \varepsilon\mu_\alpha(k,t)\int_{t}^{\infty} \frac{1}{s^{3/2}}\ ds \leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t),$$

and (127) follows for $i=5$. Next, for u_6 we have that

$$\begin{aligned}\left|\frac{\Lambda_-}{\Lambda_0}\int_{t}^{\infty} e^{\Lambda_+(t-s)} q_1(\mathbf{k},s)\ ds\right| &\leq \frac{\varepsilon}{t^{3/2}}\frac{|\Lambda_-|}{\Lambda_0}\mu_\alpha(k,t)\int_{t}^{\infty} e^{\Lambda_+(t-s)}\ ds\\ &\leq \frac{\varepsilon}{t^{3/2}}\frac{|\Lambda_-|}{\Lambda_0\Lambda_+}\mu_\alpha(k,t),\end{aligned}$$

and (127) follows for $i=6$. The bound (128) for $i=7$ and the bound (127) for $i=8$ are obtained exactly as the bounds on u_4 and u_5. To bound u_9 we split the corresponding integral into two parts. We have that

$$\begin{aligned}\left|\frac{1}{\Lambda_0}\int_{1}^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}\mathbf{q}_0(\mathbf{k},s)\ ds\right| &\leq \frac{1}{\Lambda_0}\int_{1}^{\frac{t+1}{2}} e^{\Lambda_-(t-s)}\frac{\varepsilon}{s^{3/2}}\mu_\alpha(k,s)\ ds\\ &\leq \varepsilon\frac{1}{\Lambda_0}e^{\Lambda_-\frac{t-1}{2}}\mu_\alpha(k,1)\int_{1}^{\frac{t+1}{2}}\frac{1}{s^{3/2}}\ ds\\ &\leq \varepsilon\mu_{\alpha+1}(k,t),\end{aligned}$$

and that

$$\begin{aligned}\left|\frac{1}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\mathbf{q}_0(\mathbf{k},s)\ ds\right| &\leq \frac{1}{\Lambda_0}\int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)}\frac{\varepsilon}{s^{3/2}}\mu_\alpha(k,s)\ ds\\ &\leq \varepsilon\frac{1}{\Lambda_0}\mu_\alpha(k,t)\int_{\frac{t+1}{2}}^{t}\frac{ds}{s^{3/2}} \leq \frac{\varepsilon}{t^{1/2}}\frac{1}{\Lambda_0}\mu_\alpha(k,t)\\ &\leq \varepsilon\mu_{\alpha+1}(k,t),\end{aligned}$$

and therefore we find using the triangle inequality that

$$\left|\frac{1}{\Lambda_0}\int_{1}^{t} e^{\Lambda_-(t-s)}\mathbf{q}_0(\mathbf{k},s)\ ds\right| \leq \varepsilon\mu_{\alpha+1}(k,t). \tag{129}$$

The bound (127) now follows for $i=9$, since

$$|u_9(\mathbf{k},t)| \leq \varepsilon k\mu_{\alpha+1}(k,t) \leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t).$$

Finally, since $u_{10} = -i\mathbf{k}^T/\Lambda_+\,(\tau_5)^\perp$ and since $\left|-i\mathbf{k}^T/\Lambda_+\right| \leq$ const., the bound (127) follows for $i=10$ from the bound (125) for $i=5$. This completes the proof of Proposition 9.

11.3. Bounds on $\mathbf{v_1}$ and $\mathbf{v_2}$

We write $\mathbf{v}_1 = \sum_{i=1}^{6} \mathbf{v}_{1,i}$, with $\mathbf{v}_{1,i}$ the i-th term in (82), and we bound each of the terms individually. The inequality (95) then follows using the triangle inequality.

Proposition 10. *For all $\alpha \geq 0$ and $i = 1, \ldots, 6$ we have the bound*

$$|\mathbf{v}_{1,i}(\mathbf{k}, t)| \leq \varepsilon \bar{\mu}_{\alpha+1}(k, t) + \sigma_i \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k, t), \tag{130}$$

with $\sigma_2 = 0$ and $\sigma_i = 1$ otherwise, uniformly in $\mathbf{k} \in \mathbf{R}^2$ and $t \geq 1$.

Proof. For $i = 1$ inequality (130) follows from (93). Next, since $\mathbf{v}_{1,2}(\mathbf{k}, t) = u_2(\mathbf{k}, t) i\mathbf{k}/k$ we find that

$$|\mathbf{v}_{1,2}(\mathbf{k}, t)| \leq |u_2(\mathbf{k}, t)|,$$

and therefore (130) follows for $i = 2$ from (128). Similarly, for $i = 3, \ldots, 6$ the bound (130) follows from (127) and (128), since $\mathbf{v}_{1,3}(\mathbf{k}, t) = i\frac{\mathbf{k}}{k}\, u_4(\mathbf{k}, t)$, $\mathbf{v}_{1,4}(\mathbf{k}, t) = -i\frac{\mathbf{k}}{k}\, u_5(\mathbf{k}, t)$, $\mathbf{v}_{1,5}(\mathbf{k}, t) = i\frac{\mathbf{k}}{k}\, u_7(\mathbf{k}, t)$ and $\mathbf{v}_{1,6}(\mathbf{k}, t) = -i\frac{\mathbf{k}}{k}\, u_8(\mathbf{k}, t)$. □

For $\mathbf{v}_2$ we write $\mathbf{v}_2 = \sum_{i=1}^{3} \mathbf{v}_{2,i}$, with $\mathbf{v}_{2,i}$ the i-th term in (83), and we bound each of the terms individually. The inequality (96) then follows using the triangle inequality.

Proposition 11. *For all $\alpha \geq 0$ and $i = 1, \ldots, 3$ we have the bounds*

$$|\mathbf{v}_{2,i}(\mathbf{k}, t)| \leq \frac{\varepsilon}{t^{\sigma_i}} \mu_{\alpha+1}(k, t), \tag{131}$$

with $\sigma_1 = \sigma_2 = 0$ and $\sigma_3 = 1$, uniformly in $\mathbf{k} \in \mathbf{R}^2$ and $t \geq 1$.

Proof. For $\mathbf{v}_{2,1}$ we have, using Proposition 15, that

$$\left| \frac{-1}{\Lambda_-} P_2 \left(-\Lambda_-\ P_1 \tau^*_{-,2}(\mathbf{k}) \right)^\perp e^{\Lambda_-(t-1)} \right| \leq \varepsilon \mu_{\alpha+1}(k, 1) e^{\Lambda_-(t-1)} \leq \varepsilon \mu_{\alpha+1}(k, t),$$

and (131) follows for $i = 1$. For $i = 2$ the bound (131) follows from (129). Finally, since $\mathbf{v}_{2,3} = -P_2 (\tau_5)^\perp / \Lambda_+$ we get, using (125) for $i = 5$, that

$$|\mathbf{v}_{2,3}(\mathbf{k}, t)| \leq \frac{1}{\Lambda_+} \frac{\varepsilon}{t^{3/2}} \mu_\alpha(k, t) \leq \frac{\varepsilon}{t} \mu_{\alpha+1}(k, t),$$

and the bound (131) follows for $i = 3$. □

Remark. The bounds on $\mathbf{v}_1$ that are proved in this section are sufficiently strong for proving Theorem 2 of Section 5 (existence of a solution). For the proof of Theorem 7 of Section 8 (asymptotic behavior of the solution) more detailed bounds are necessary. These bounds are stated in Proposition 12 (see Appendix III, Section 12.2). A proof of Proposition 12 is the content of Appendix IV.

12. Appendix III

In this appendix we prove (111), (112) and (113).

12.1. Asymptotic behavior of u

Let

$$U(\mathbf{k},t)=\Lambda_+\left(-\tau^*_{-,1}(\mathbf{k})+\frac{1}{\Lambda_0}\int_1^t q_1(\mathbf{k},s)\;ds\right)e^{\Lambda_-(t-1)}.$$

Using the triangle inequality we get that

$$|u(\mathbf{k},t)-u_{as}(k,t)|\leq\left|U(\mathbf{k},t)-c\;e^{-k^2t}\right|+\left|c\;e^{-k^2t}-u_{as}(\mathbf{k},t)\right|+|u(k,t)-U(k,t)|\,. \tag{132}$$

We bound each term in (132) separately. First, we have that

$$\lim_{t\to\infty}U\left(\frac{\mathbf{k}}{t^{1/2}},t\right)=c\;e^{-k^2}, \tag{133}$$

and furthermore that

$$|U(\mathbf{k},t)|\leq\left(\varepsilon\mu_\alpha(k,1)+\varepsilon\mu_\alpha(k,1)\int_1^\infty\frac{1}{s^{3/2}}ds\right)e^{\Lambda_-(t-1)}\leq\varepsilon\mu_\alpha(k,t),$$

so that

$$\left|U(\frac{\mathbf{k}}{t^{1/2}},t)\right|\leq\varepsilon\mu_\alpha(k,1). \tag{134}$$

From (133) and (134) it follows by the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty}t\int_{\mathbf{R}^2}\left|U(\mathbf{k},t)-c\;e^{-k^2t}\right|\;d^2\mathbf{k}=\lim_{t\to\infty}\int_{\mathbf{R}^2}\left|U(\frac{\mathbf{k}}{t^{1/2}},t)-c\;e^{-k^2}\right|\;d^2\mathbf{k}=0,$$

as required. Next

$$\begin{aligned}\lim_{t\to\infty}t\int_{\mathbf{R}^2}\left|u_{as}(\mathbf{k},t)-c\;e^{-k^2t}\right|\;d^2\mathbf{k}\;&\leq\lim_{t\to\infty}t\int_{\mathbf{R}^2}\left(\;d+\frac{i\mathbf{k}^T}{k}\;\mathbf{b}\right)e^{-kt}\;d^2\mathbf{k}\\&\leq\lim_{t\to\infty}t\int_{\mathbf{R}^2}(|d|+|\mathbf{b}|)\,e^{-kt}\;d^2\mathbf{k}\\&=\lim_{t\to\infty}t\;\;(|d|+|\mathbf{b}|)\,\frac{2\pi}{t^2}=0,\end{aligned}$$

as required. For the last term in (132) we have, writing as in Appendix II $u=\sum_{i=1}^{10}u_i$, with u_i the i-th term in (81),

$$u(\mathbf{k},t)-U(\mathbf{k},t)=u_2(\mathbf{k},t)+\frac{\Lambda_+}{\Lambda_0}\int_1^t\left(e^{\Lambda_-(t-s)}-e^{\Lambda_-(t-1)}\right)q_1(\mathbf{k},s)ds+\sum_{i=4\ldots10}u_i(\mathbf{k},t). \tag{135}$$

Since

$$\int_1^{\frac{t+1}{2}} e^{\Lambda_-(t-s)} |\Lambda_-| (s-1) \frac{\varepsilon}{s^{3/2}} \mu_\alpha(k,s)\ ds \leq \varepsilon \mu_\alpha(k,1) e^{\Lambda_- \frac{t-1}{2}} |\Lambda_-| \left(\frac{t-1}{t}\right)^2 t^{1/2}$$
$$\leq \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t), \tag{136}$$

and

$$\int_{\frac{t+1}{2}}^t e^{\Lambda_-(t-s)} |\Lambda_-| (s-1) \frac{\varepsilon}{s^{3/2}} \mu_\alpha(k,s)\ ds \leq \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t) \int_{\frac{t+1}{2}}^t e^{\Lambda_-(t-s)} |\Lambda_-|\ ds$$
$$\leq \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t), \tag{137}$$

we find, using that $1 - e^x \leq -x$ for all $x \leq 0$, that

$$\begin{aligned}
&\left| \frac{\Lambda_+}{\Lambda_0} \int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)} \right) q_1(\mathbf{k},s)\ ds \right| \\
&\leq \int_1^t e^{\Lambda_-(t-s)} \left(1 - e^{\Lambda_-(s-1)}\right) \frac{\varepsilon}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
&\leq - \int_1^t e^{\Lambda_-(t-s)} \Lambda_-(s-1) \frac{\varepsilon}{s^{3/2}} \mu_\alpha(k,s)\ ds \leq \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t),
\end{aligned} \tag{138}$$

and therefore we find from (135) using the triangle inequality and using Proposition 9 that

$$\begin{aligned}
|u(\mathbf{k},t) - U(\mathbf{k},t))| &\leq |u_2(\mathbf{k},t)| + \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t) + \sum_{i=4\ldots 10} |u_i(\mathbf{k},t)| \\
&\leq \varepsilon \bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\lim_{t\to\infty} t \int_{\mathbf{R}^2} |u(\mathbf{k},t) - U(\mathbf{k},t)|\ dk \\
&\leq \lim_{t\to\infty} t \int_{\mathbf{R}^2} \left(\varepsilon \bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha(k,t) \right)\ d^2\mathbf{k} \leq \lim_{t\to\infty} t \left(\frac{\varepsilon}{t^2} + \frac{\varepsilon}{t^{3/2}} \right) = 0,
\end{aligned}$$

as required. This completes the proof of (111).

12.2. Asymptotic behavior of $\mathbf{v}_1$

Let

$$\mathbf{V}_1(\mathbf{k},t) = \left(-\tau^*_{-,1}(\mathbf{k}) + \frac{1}{\Lambda_0} \int_1^t q_1(\mathbf{k},s)\ ds \right)\ i\mathbf{k} e^{\Lambda_-(t-1)}.$$

Using the triangle inequality we get that

$$|\mathbf{v}_1(\mathbf{k},t) - \mathbf{v}_{1,as}(\mathbf{k},t)| \leq \left| \mathbf{V}_1(\mathbf{k},t) - c\ i\mathbf{k} e^{-k^2 t} \right| + \left| c\ i\mathbf{k} e^{-k^2 t} - \mathbf{v}_{1,as}(\mathbf{k},t) \right|$$

$$+\left|\mathbf{v}_1(\mathbf{k},t)-\mathbf{V}_1(\mathbf{k},t)\right|. \tag{139}$$

We bound each term in (139) separately. First, we have that

$$\lim_{t\to\infty} t^{1/2}\mathbf{V}_1(\frac{\mathbf{k}}{t^{1/2}},t)=c\ i\mathbf{k}e^{-k^2}, \tag{140}$$

and furthermore that

$$|\mathbf{V}_1(\mathbf{k},t)|\le\left(\varepsilon\mu_{\alpha+1}(k,1)+\varepsilon\mu_{\alpha+1}(k,1)\int_1^\infty\frac{1}{s^{3/2}}ds\right)ke^{\Lambda_-(t-1)}\le\varepsilon k\mu_{\alpha+1}(k,t),$$

so that

$$\left|t^{1/2}\mathbf{V}_1(\frac{\mathbf{k}}{t^{1/2}},t)\right|\le\varepsilon k\mu_{\alpha+1}(k,1). \tag{141}$$

From (140) and (141) it follows by the Lebesgue dominated convergence theorem that

$$\begin{aligned}&\lim_{t\to\infty}t^{3/2}\int_{\mathbf{R}^2}\left|\mathbf{V}_1(\mathbf{k},t)-c\ i\mathbf{k}e^{-k^2t}\right|\ d^2\mathbf{k}\\&=\lim_{t\to\infty}\int_{\mathbf{R}^2}\left|t^{1/2}\mathbf{V}_1(\frac{\mathbf{k}}{t^{1/2}},t)-c\ i\mathbf{k}e^{-k^2}\right|\ d^2\mathbf{k}=0,\end{aligned}$$

as required. Next

$$\begin{aligned}\lim_{t\to\infty}t^{3/2}\int_{\mathbf{R}^2}\left|\mathbf{v}_{1,as}(\mathbf{k},t)-c\ i\mathbf{k}e^{-k^2}\right|d^2\mathbf{k}&\le\lim_{t\to\infty}t^{3/2}\int_{\mathbf{R}^2}\left(\frac{i}{k}\mathbf{k}d-P_1\mathbf{b}\right)e^{-kt}d^2\mathbf{k}\\&\le\lim_{t\to\infty}t^{3/2}\int_{\mathbf{R}^2}\left(|d|+|\mathbf{b}|\right)e^{-kt}\ d^2\mathbf{k}\\&=\lim_{t\to\infty}t^{3/2}\left(|d|+|\mathbf{b}|\right)\frac{2\pi}{t^2}=0,\end{aligned}$$

as required. Finally, for the last term in (139) we have the following proposition:

Proposition 12. *Let* $\mathbf{v}_1$ *and* $\mathbf{V}_1$ *be as defined above. Then,*

$$|\mathbf{v}_1(\mathbf{k},t)-\mathbf{V}_1(\mathbf{k},t)|\le\frac{\varepsilon}{t^{2/3}}\mu_\alpha(k,t)+\frac{\varepsilon}{t^{1/3}}\mu_\alpha^{5/6}(k,t)+\frac{\varepsilon}{t^{1/2}}\mu_\alpha^{3/4}(k,t)+\varepsilon\bar{\mu}_{\alpha+1}(k,t). \tag{142}$$

See Appendix IV for a proof.

From Proposition 12 it follows that

$$\begin{aligned}\lim_{t\to\infty}t^{3/2}\int_{\mathbf{R}^2}|\mathbf{v}_1(\mathbf{k},t)-\mathbf{V}_1(\mathbf{k},t)|\,d^2\mathbf{k}&\le\lim_{t\to\infty}t^{3/2}\int_{\mathbf{R}^2}\Big(\frac{\varepsilon}{t^{2/3}}\mu_\alpha(k,t)+\frac{\varepsilon}{t^{1/3}}\mu_\alpha^{5/6}(k,t)\\&+\frac{\varepsilon}{t^{1/2}}\mu_\alpha^{3/4}(k,t)+\varepsilon\bar{\mu}_{\alpha+1}(k,t)\Big)d^2\mathbf{k}\le\lim_{t\to\infty}t^{3/2}\left(\frac{\varepsilon}{t^{5/3}}+\frac{\varepsilon}{t^2}\right)=0,\end{aligned}$$

as required. This completes the proof of (112).

12.3. Asymptotic behavior of $\mathbf{v}_2$

Let

$$\mathbf{V}_2(\mathbf{k},t) = P_2\left(\tau^*_{-,2}(\mathbf{k}) + \frac{1}{\Lambda_0}\int_1^t \mathbf{q}_0(\mathbf{k},s)\ ds\right)^\perp e^{\Lambda_-(t-1)}.$$

Using the triangle inequality we get that

$$|\mathbf{v}_2(\mathbf{k},t) - \mathbf{v}_{2,as}(\mathbf{k},t)| \leq \left|\mathbf{V}_2(\mathbf{k},t) - P_2 e^{-k^2 t}\ \mathbf{a}\right| + |\mathbf{v}_2(\mathbf{k},t) - \mathbf{V}_2(\mathbf{k},t)|\,. \tag{143}$$

We bound each term in (143) separately. First, we have that

$$\lim_{t\to\infty} \mathbf{V}_2(\frac{\mathbf{k}}{t^{1/2}},t) = P_2 e^{-k^2}\mathbf{a}, \tag{144}$$

and furthermore that

$$|\mathbf{V}_2(\mathbf{k},t)| \leq \left(\varepsilon\mu_{\alpha+1}(k,1) + \varepsilon\mu_{\alpha+1}(k,1)\int_1^\infty \frac{1}{s^{3/2}}ds\right) e^{\Lambda_-(t-1)} \leq \varepsilon\mu_{\alpha+1}(k,t),$$

so that

$$\left|\mathbf{V}_2(\frac{\mathbf{k}}{t^{1/2}},t)\right| \leq \varepsilon\mu_{\alpha+1}(k,1). \tag{145}$$

From (144) and (145) it follows by the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} t\int_{\mathbf{R}^2}\left|\mathbf{V}_2(\mathbf{k},t) - P_2 e^{-k^2 t}\mathbf{a}\right|\ d^2\mathbf{k} = \lim_{t\to\infty}\int_{\mathbf{R}^2}\left|\mathbf{V}_2(\frac{\mathbf{k}}{t^{1/2}},t) - P_2 e^{-k^2}\mathbf{a}\right|\ d^2\mathbf{k} = 0,$$

as required. For the second term in (143) we have, writing as in Appendix II $\mathbf{v}_2 = \sum_{i=1}^3 \mathbf{v}_{2,i}$, with $\mathbf{v}_{2,i}$ the i-th term in (83),

$$\mathbf{v}_2(\mathbf{k},t) - \mathbf{V}_2(\mathbf{k},t) = \frac{1}{\Lambda_0}\int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right) P_2\mathbf{q}_0(\mathbf{k},s)^\perp\ ds + \mathbf{v}_{2,3}(\mathbf{k},t). \tag{146}$$

Using (136) and (137) and that $1-e^x \leq -x$ for all $x\leq 0$, we find as in (138) that

$$\begin{aligned}&\left|\frac{1}{\Lambda_0}\int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right) P_2\mathbf{q}_0(\mathbf{k},s)^\perp\ ds\right|\\ &\leq \frac{1}{\Lambda_0}\int_1^t e^{\Lambda_-(t-s)}\left(1 - e^{\Lambda_-(s-1)}\right)\frac{\varepsilon}{s^{3/2}}\mu_\alpha(k,s)\ ds \leq \frac{\varepsilon}{t^{1/2}}\frac{1}{\Lambda_0}\mu_\alpha(k,t),\end{aligned}$$

and therefore we find from (146) using the triangle inequality and using (131) for $i=3$ that

$$|\mathbf{v}_2(\mathbf{k},t) - \mathbf{V}_2(\mathbf{k},t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t) + |\mathbf{v}_{2,3}(\mathbf{k},t)| \leq \frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t),$$

from which it follows that

$$\lim_{t\to\infty} t\int_{\mathbf{R}^2}|\mathbf{v}_2(\mathbf{k},t) - \mathbf{V}_2(\mathbf{k},t)|\ d^2\mathbf{k} \leq \lim_{t\to\infty} t\int_{\mathbf{R}^2}\frac{\varepsilon}{t^{1/2}}\mu_\alpha(k,t)\ d^2\mathbf{k} \leq \lim_{t\to\infty} t\frac{\varepsilon}{t^{3/2}} = 0,$$

as required. This completes the proof of (113).

13. Appendix IV

In this appendix we prove Proposition 12 (see Appendix III, Section 12.2). This proposition is a strengthened version of Proposition 10 (see Appendix II, Section 11.3). The proof is rather lengthy and we therefore split it in several pieces. We start by proving some general bounds.

13.1. Three inequalities

Proposition 13. *Let $\alpha \geq 0$. Then,*

$$\int_1^t \left(e^{-k(t-s)} - e^{-k(t-1)}\right) \mu_\alpha(k,s) \frac{ds}{s^{3/2}} \leq \text{const.} \left(\frac{1}{t^{2/3}} \mu_\alpha(k,t) + \frac{1}{t^{1/3}} \mu_\alpha^{5/6}(k,t)\right) \tag{147}$$

$$\int_t^\infty e^{k(t-s)} \mu_\alpha(k,s) \frac{ds}{s^{3/2}} \leq \text{const.} \left(\frac{1}{t^{2/3}} \mu_\alpha(k,t) + \frac{1}{t^{1/2}} \mu_\alpha^{3/4}(k,t)\right) \tag{148}$$

$$\frac{k}{\Lambda_0} \int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right) \mu_\alpha(k,s) \frac{ds}{s^{3/2}} \leq \text{const.} \frac{1}{t} \mu_\alpha(k,t), \tag{149}$$

uniformly in $\mathbf{k} \in \mathbf{R}^2$ and $t \geq 1$.

We first prove (147) for $1 \leq t \leq 2$. We have

$$\begin{aligned}
&\int_1^t \left(e^{-k(t-s)} - e^{-k(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\; ds \\
&\leq \text{const. } \mu_\alpha(k,1) \int_1^t \frac{ds}{s^{3/2}} \leq \text{const. } \mu_\alpha(k,1) \leq \frac{\text{const.}}{t^{2/3}} \mu_\alpha(k,t),
\end{aligned}$$

as required. For $t > 2$ we split the integral in (147) into two. For the first part we have

$$\begin{aligned}
&\int_1^{t-(t-1)^{5/6}} \left(e^{-k(t-s)} - e^{-k(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\; ds \\
&\leq \text{const. } \mu_\alpha(k,1) \int_1^{t-(t-1)^{5/6}} \left(e^{-k(t-s)} - e^{-k(t-1)}\right) \frac{1}{s^{3/2}}\; ds \\
&\leq \text{const. } \mu_\alpha(k,1) \int_1^{t-(t-1)^{5/6}} e^{-k(t-s)} \left(1 - e^{-k(s-1)}\right) \frac{1}{s^{3/2}}\; ds \\
&\leq \text{const. } \mu_\alpha(k,1) e^{-k(t-1)^{5/6}} \int_1^{t-(t-1)^{5/6}} (s-1)k\; \frac{1}{s^{3/2}}\; ds \\
&\leq \text{const. } t^{1/2} \mu_\alpha(k,1) e^{-k(t-1)^{5/6}} k \left(\frac{t-1}{t}\right)^2 \leq \frac{\text{const.}}{t^{1/3}} \mu_\alpha^{5/6}(k,t),
\end{aligned}$$

as required, and for the other part we get,

$$\begin{aligned}
&\int_{t-(t-1)^{5/6}}^{t} \left(e^{-k(t-s)} - e^{-k(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
&\leq \frac{\text{const.}}{t^{3/2}} \mu_\alpha(k,t) \int_{t-(t-1)^{5/6}}^{t} ds \leq \frac{\text{const.}}{t^{3/2}} \mu_\alpha(k,t) t^{5/6} \leq \frac{\text{const.}}{t^{2/3}} \mu_\alpha(k,t),
\end{aligned}$$

as required. We now prove (148). Namely,

$$\begin{aligned}
&\int_{t}^{\infty} e^{k(t-s)} \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
&\leq \text{const. } \mu_\alpha(k,t) \left(\int_{t}^{t+t^{3/4}} e^{k(t-s)} \frac{1}{s^{3/2}}\ ds + \int_{t+t^{3/4}}^{\infty} e^{k(t-s)} \frac{1}{s^{3/2}}\ ds \right) \\
&\leq \text{const. } \mu_\alpha(k,t) \left(\int_{t}^{t+t^{3/4}} \frac{1}{s^{3/2}}\ ds + e^{-kt^{3/4}} \int_{t+t^{3/4}}^{\infty} \frac{1}{s^{3/2}}\ ds \right) \\
&\leq \text{const. } \mu_\alpha(k,t) \left(\frac{1}{t^{3/4}} + \frac{1}{t^{1/2}} e^{-kt^{3/4}} \right) \leq \frac{\text{const.}}{t^{3/4}} \mu_\alpha(k,t) + \frac{\text{const.}}{t^{1/2}} \mu_\alpha^{3/4}(k,t),
\end{aligned}$$

and (148) follows. We finally prove (149). We have that

$$\begin{aligned}
&\frac{k}{\Lambda_0} \int_{1}^{t} \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
&\leq \text{const. } \mu_{\alpha+1/2}(k,1) \int_{1}^{\frac{t+1}{2}} \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right) |\Lambda_-|^{1/2} \frac{1}{s^{3/2}}\ ds \\
&\leq \text{const. } \mu_{\alpha+1/2}(k,1) \int_{1}^{\frac{t+1}{2}} e^{\Lambda_-(t-s)} \left(1 - e^{\Lambda_-(s-1)}\right) |\Lambda_-|^{1/2} \frac{1}{s^{3/2}}\ ds \\
&\leq \text{const. } \mu_{\alpha+1/2}(k,1) e^{\Lambda_- \frac{t-1}{2}} \int_{1}^{\frac{t+1}{2}} (s-1) |\Lambda_-|^{3/2} \frac{1}{s^{3/2}}\ ds \\
&\leq \text{const. } t^{1/2} \mu_{\alpha+1/2}(k,1) e^{\Lambda_- \frac{t-1}{2}} |\Lambda_-|^{3/2} \left(\frac{t-1}{t}\right)^2 \leq \frac{\text{const.}}{t} \mu_{\alpha+1/2}(k,t),
\end{aligned}$$

and since $\left|e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right| \leq e^{\Lambda_-(t-s)}$ we have that

$$\begin{aligned}
&\frac{k}{\Lambda_0} \int_{\frac{t+1}{2}}^{t} \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)}\right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
&\leq \frac{\text{const.}}{t^{3/2}}\ \frac{\Lambda_+^{1/2}}{\Lambda_0} \mu_\alpha(k,t) \int_{\frac{t+1}{2}}^{t} e^{\Lambda_-(t-s)} |\Lambda_-|^{1/2}\ ds \\
&\leq \frac{\text{const.}}{t^{3/2}} \mu_\alpha(k,t) \int_{\frac{t+1}{2}}^{t} \frac{1}{\sqrt{t-s}}\ ds \leq \frac{\text{const.}}{t} \mu_\alpha(k,t),
\end{aligned}$$

and (149) now follows using the triangle inequality. This completes the proof of Proposition 13.

13.2. Proof of Proposition 12

Let $\mathbf{v}_D = \mathbf{v}_1 - \mathbf{V}_1$. Using the definitions and writing as in Appendix II $\mathbf{v}_1 = \sum_{i=1}^{6} \mathbf{v}_{1,i}$, with $\mathbf{v}_{1,i}$ the i-th term in (82) and $\tau = \sum_{i=1}^{5} \tau_i$, with τ_i the i-th term in (80), we find that

$$\mathbf{v}_D(\mathbf{k},t) = \frac{i\mathbf{k}}{\Lambda_0} \int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)} \right) q_1(\mathbf{k},s)\, ds + \sum_{i=3}^{5} P_1 \tau_i(\mathbf{k},t)^\perp + \sum_{i=2}^{6} \mathbf{v}_{1,i}(\mathbf{k},t). \tag{150}$$

We write $\mathbf{v}_D = \sum_{i=1}^{3} \mathbf{v}_{D,i}$, with $\mathbf{v}_{D,i}$ the i-th of the three terms in (150), and we now bound each term individually. The inequality (142) then follows using the triangle inequality.

First, using (149) we find for $\mathbf{v}_{D,1}$ that

$$\begin{aligned} &\left| \frac{i\mathbf{k}}{\Lambda_0} \int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)} \right) q_1(\mathbf{k},s)\, ds \right| \\ &\leq \frac{k}{\Lambda_0} \int_1^t \left(e^{\Lambda_-(t-s)} - e^{\Lambda_-(t-1)} \right) \frac{\varepsilon}{s^{3/2}} \mu_\alpha(k,s)\, ds \leq \frac{\varepsilon}{t} \mu_\alpha(k,t), \end{aligned}$$

as required. Next using (125) of Proposition 8 for $i = 3, \dots, 5$, we find for $\mathbf{v}_{D,2}$ that

$$\left| \sum_{i=3}^{5} P_1 \tau_i(\mathbf{k},t)^\perp \right| \leq \sum_{i=3}^{5} |\tau_i(\mathbf{k},t)| \leq \frac{\varepsilon}{t} \mu_\alpha(k,t),$$

as required. This leaves us with proving an improved version of Proposition 10.

Proposition 14. *For all $\alpha \geq 0$ and $i = 2, \dots, 6$ we have the bound*

$$|\mathbf{v}_{1,i}(\mathbf{k},t)| \leq \varepsilon \bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{2/3}} \mu_\alpha(k,t) + \frac{\varepsilon}{t^{1/2}} \mu_\alpha^{3/4}(k,t) + \frac{\varepsilon}{t^{1/3}} \mu_\alpha^{5/6}(k,t), \tag{151}$$

uniformly in $\mathbf{k} \in \mathbf{R}^2$ and $t \geq 1$.

For $i = 2$ the bound (151) has already been proved in Proposition 10. We now prove the bound (151) for $i = 3$ and $i = 5$. We have that

$$\begin{aligned} \left| -\frac{1}{2} \frac{i}{k} \mathbf{k} \int_1^t e^{-k(t-s)} q_1(\mathbf{k},s)\, ds \right| &\leq \varepsilon \int_1^t e^{-k(t-s)} \frac{1}{s^{3/2}} \mu_\alpha(k,s)\, ds, \\ \left| -\frac{1}{2} \int_1^t e^{-k(t-s)} P_1 \mathbf{q}_0(\mathbf{k},s)^\perp\, ds \right| &\leq \varepsilon \int_1^t e^{-k(t-s)} \frac{1}{s^{3/2}} \mu_\alpha(k,s)\, ds, \end{aligned}$$

which proves the bounds, since by Proposition 15 (see Appendix V)

$$\varepsilon e^{-k(t-1)} \int_1^t \frac{1}{s^{3/2}} \mu_\alpha(k,s)\, ds \leq \varepsilon e^{-k(t-1)} \mu_\alpha(k,1) \left(\frac{t-1}{t} \right) \leq \varepsilon \bar{\mu}_{\alpha+1}(k,t),$$

and therefore, and using (147),

$$
\begin{aligned}
&\varepsilon \int_1^t e^{-k(t-s)} \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
&\leq \varepsilon e^{-k(t-1)} \int_1^t \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds + \varepsilon \int_1^t \left(e^{-k(t-s)} - e^{-k(t-1)} \right) \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds \\
&\leq \varepsilon \bar{\mu}_{\alpha+1}(k,t) + \frac{\varepsilon}{t^{2/3}} \mu_\alpha(k,t) + \frac{\varepsilon}{t^{1/3}} \mu_\alpha^{5/6}(k,t),
\end{aligned}
$$

as required. Finally, for $i = 4$ and $i = 6$, the bound (151) is an immediate consequence of (148), since

$$
\begin{aligned}
\left| -\frac{1}{2} \frac{i}{k} \mathbf{k} \int_t^\infty e^{k(t-s)} q_1(\mathbf{k},s)\ ds \right| &\leq \varepsilon \int_t^\infty e^{k(t-s)} \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds, \\
\left| \frac{1}{2} \int_t^\infty e^{k(t-s)} P_1 \mathbf{q}_0(\mathbf{k},s)^\perp\ ds \right| &\leq \varepsilon \int_t^\infty e^{k(t-s)} \frac{1}{s^{3/2}} \mu_\alpha(k,s)\ ds.
\end{aligned}
$$

This completes the proof of Proposition 12.

14. Appendix V

14.1. Main technical lemma

Proposition 15. *Let $\alpha' \geq \beta' \geq \gamma' \geq 0$ and $\mu > 0$. Then, we have the bound*

$$
\frac{1}{1+k^{\alpha'}} e^{\mu \Lambda_-(t-1)} \left| \Lambda_- \right|^{\beta'} \left(\frac{t-1}{t} \right)^{\gamma'} \leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1 + \left(k t^{1/2} \right)^{\alpha'-\beta'+\gamma'}}, \tag{152}
$$

uniformly in $\mathbf{k} \in \mathbf{R}^2$ and $t \geq 1$. Similarly, for positive α', β', γ' with $\alpha' - \beta' + \gamma' \geq 0$ and $\mu > 0$ we have the bound

$$
\frac{1}{1+k^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'} \left(\frac{t-1}{t} \right)^{\gamma'} \leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1 + (kt)^{\alpha'-\beta'+\gamma'}}, \tag{153}
$$

uniformly in $\mathbf{k} \in \mathbf{R}^2$ and $t \geq 1$.

Proof. We first prove (152). For $1 \leq t \leq 2$ we have that

$$
\begin{aligned}
&\frac{1}{1+k^{\alpha'}} e^{\mu \Lambda_-(t-1)} \left| \Lambda_- \right|^{\beta'} \left(\frac{t-1}{t} \right)^{\gamma'} \\
&\leq \text{const.}\ \frac{1}{1+k^{\alpha'}} e^{\mu \Lambda_-(t-1)} \left| \Lambda_-\,(t-1) \right|^{\gamma'}\ \ \left| \Lambda_- \right|^{\beta'-\gamma'} \\
&\leq \text{const.}\ \frac{1}{1+k^{\alpha'}}\ \ \left| \Lambda_- \right|^{\beta'-\gamma'} \leq \text{const.}\ \frac{1}{1+k^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const.}\ \frac{1}{t^{\beta'}} \frac{1}{1 + \left(k t^{1/2} \right)^{\alpha'-\beta'+\gamma'}},
\end{aligned}
$$

as claimed, and for $t > 2$ we use that

$$
\begin{aligned}
&\left(1+\left(kt^{1/2}\right)^{\alpha'-\beta'+\gamma'}\right) e^{\mu\Lambda_-(t-1)}\left|\Lambda_- t\right|^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.}\left(1+\left(kt^{1/2}\right)^{\alpha'}\right) e^{\frac{1}{2}\mu\Lambda_- t}\left|\Lambda_- t\right|^{\beta'} \\
&\leq \text{const.}\left(1+\frac{k^{\alpha'}}{\left|\Lambda_-\right|^{\alpha'/2}}\left|\Lambda_- t\right|^{\alpha'/2}\left|\Lambda_- t\right|^{\beta'} e^{\frac{1}{2}\mu\Lambda_- t}\right) \\
&\leq \text{const.}\left(1+\frac{k^{\alpha'}}{\left|\Lambda_-\right|^{\alpha'/2}}\right) \leq \text{const.}\left(1+k^{\alpha'/2}\right) \leq \text{const.}\left(1+k^{\alpha'}\right),
\end{aligned}
$$

and (152) follows. We now prove (153). For $1 \leq t \leq 2$ and $k \leq 1$ we have that

$$
\frac{1}{1+k^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'} \leq \text{const.} \leq \text{const.}\ \frac{1}{t^{\beta'}}\frac{1}{1+(kt)^{\alpha'-\beta'+\gamma'}},
$$

and for $1 \leq t \leq 2$ and $k > 1$ we have that

$$
\begin{aligned}
\frac{1}{1+k^{\alpha'}} e^{-\mu k(t-1)} k^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'} &\leq \text{const.}\ \frac{1}{1+k^{\alpha'}} e^{-\mu k(t-1)}\left(k\left(t-1\right)\right)^{\gamma'}\ k^{\beta'-\gamma'} \\
&\leq \text{const.}\ \frac{1}{1+k^{\alpha'}}\ k^{\beta'-\gamma'} \leq \text{const.}\ \frac{1}{1+k^{\alpha'-\beta'+\gamma'}} \\
&\leq \text{const.}\ \frac{1}{t^{\beta'}}\frac{1}{1+(kt)^{\alpha'-\beta'+\gamma'}}.
\end{aligned}
$$

Finally, for $t > 2$ we use that

$$
\begin{aligned}
&\left(1+(kt)^{\alpha'-\beta'+\gamma'}\right) e^{-\mu k(t-1)}\,(kt)^{\beta'}\left(\frac{t-1}{t}\right)^{\gamma'} \\
&\leq \text{const.}\left(1+(kt)^{\alpha'-\beta'+\gamma'}\right) e^{-\frac{1}{2}\mu kt}\,(kt)^{\beta'} \leq \text{const.} \leq \text{const.}\left(1+k^{\alpha'}\right),
\end{aligned}
$$

and (153) follows. □

14.2. Bound on convolution

Proposition 16. *Let $\alpha > 2$, and let a_1 be a piecewise continuous, and a_2 be a continuous function from $\mathbf{R}^2 \times [1,\infty)$ to $\mathbf{C}$ satisfying the bounds,*

$$
|a_i(\mathbf{k},t)| \leq \mu_\alpha(k,t),
$$

$i = 1,2$. Then, the convolution $a_1 * a_2$ is a continuous function from $\mathbf{R}^2 \times [1,\infty)$ to $\mathbf{C}$ and we have the bound

$$
\left|\left(a_1 * a_2\right)(\mathbf{k},t)\right| \leq \text{const.}\ \frac{1}{t}\mu_\alpha(k,t), \tag{154}
$$

uniformly in $t \geq 1$, $\mathbf{k} \in \mathbf{R}^2$.

Proof. Continuity is elementary. We now prove (154). Let

$$D(\mathbf{k}) = \left\{\kappa \in \mathbf{R}^2 \mid |\mathbf{k} - \kappa| \leq k/2\right\}.$$

For $\mathbf{k}' \in D(\mathbf{k})$ we have that

$$k' \geq k - |\mathbf{k} - \mathbf{k}'| \geq \frac{1}{2}k.$$

Therefore we have for $a_1 * a_2$,

$$\begin{aligned}
|(a_1 * a_2)(\mathbf{k}, t)| &\leq \int_{\mathbf{R}^2 \setminus D(\mathbf{k})} \mu_\alpha(k', t)\mu_\alpha(|\mathbf{k} - \mathbf{k}'|, t)\ d^2\mathbf{k}' \\
&\quad + \int_{D(\mathbf{k})} \mu_\alpha(k', t)\mu_\alpha(|\mathbf{k} - \mathbf{k}'|, t)\ d^2\mathbf{k}' \\
&\leq \left(\sup_{\mathbf{k}' \in \mathbf{R}^2 \setminus D(\mathbf{k})} \mu_\alpha(|\mathbf{k} - \mathbf{k}'|, t)\right) \int_{\mathbf{R}^2 \setminus D(\mathbf{k})} \mu_\alpha(k', t)\ d^2\mathbf{k}' \\
&\quad + \left(\sup_{\mathbf{k}' \in D(\mathbf{k})} \mu_\alpha(k', t)\right) \int_{D(\mathbf{k})} \mu_\alpha(|\mathbf{k} - \mathbf{k}'|, t)\ d^2\mathbf{k}' \\
&\leq \text{const.}\ \mu_\alpha(k/2, t) \int_{\mathbf{R}^2} \mu_\alpha(k', t)\ d^2\mathbf{k}' \\
&\quad + \text{const.}\ \mu_\alpha(k/2, t) \int_{\mathbf{R}^2} \mu_\alpha(|\mathbf{k} - \mathbf{k}'|, t)\ d^2\mathbf{k}' \\
&\leq \text{const.}\ \frac{1}{t}\mu_\alpha(k/2, t) \leq \text{const.}\ \frac{1}{t}\mu_\alpha(k, t),
\end{aligned}$$

and (154) follows. This completes the proof of Proposition 16. □

15. Appendix VI

For the convenience of the reader we recollect in this appendix some expressions for Fourier transforms. Let $\mathbf{x} = (x, \mathbf{y}) \in \mathbf{R}^3$, with $\mathbf{y} = (y_1, y_2) \in \mathbf{R}^2$, $r = |\mathbf{x}| = \sqrt{x^2 + y^2}$, with $y = |\mathbf{y}| = \sqrt{y_1^2 + y_2^2}$, and let $\mathbf{k} = (k_1, k_2) \in \mathbf{R}^2$ with $k = \sqrt{k_1^2 + k_2^2}$. Define G by the equation

$$G(x, \mathbf{y}) = -\frac{1}{4\pi}\frac{1}{r}.$$

The function G is the Greens function of the Laplacean, *i.e.*, we have

$$\Delta G(\mathbf{x}) = \delta(\mathbf{x}),$$

and therefore

$$G(x, \mathbf{y}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbf{R}^2} e^{-i\mathbf{k}\cdot\mathbf{y}}\ \widehat{G}(\mathbf{k}, x)\ d^2\mathbf{k}, \tag{155}$$

where

$$\begin{aligned}\widehat{G}(\mathbf{k},x) &= -\frac{1}{2\pi}\int_{\mathbf{R}} e^{-ik_0x}\frac{1}{k_0^2+k^2}\ dk_0 \\ &= -\frac{1}{k}\frac{1}{2\pi}\int_{\mathbf{R}} e^{-ik_0(kx)}\frac{1}{k_0^2+1}\ dk_0 = -\frac{1}{2}\frac{1}{k}e^{-k|x|}.\end{aligned} \tag{156}$$

The vector field $\mathbf{u}_S$ of a point source is

$$\mathbf{u}_S(x,\mathbf{y}) = \nabla G(x,\mathbf{y}) = \begin{cases} \frac{1}{4\pi}\frac{x}{r^3} \\ \frac{1}{4\pi}\frac{\mathbf{y}}{r^3}, \end{cases} \tag{157}$$

or in Fourier space,

$$\widehat{\mathbf{u}_S}(\mathbf{k},x) = \begin{pmatrix}\partial_x \\ -i\mathbf{k}\end{pmatrix}\widehat{G}(\mathbf{k},x) = \begin{cases} \frac{1}{2}\mathrm{sign}(x)e^{-k|x|} \\ \frac{1}{2}\frac{i\mathbf{k}}{k}e^{-k|x|}. \end{cases} \tag{158}$$

The restriction to $x \geq 1$ of the vector field (157) and (158) multiplied by $2d$ is one of the terms in the asymptotic expressions (8)–(10) and (114)–(116), respectively. Next, let $\mathbf{e}$ be unit vector in $\mathbf{R}^2$. Define G_1 by the equation

$$\begin{aligned}G_1(\mathbf{e},x,\mathbf{y}) &= \int_x^{\mathrm{sign}(x)\infty} \nabla^{\perp}G(\xi,\mathbf{y})\cdot\mathbf{e}\ d\xi \\ &= \frac{1}{4\pi}\int_x^{\mathrm{sign}(x)\infty}\frac{\mathbf{y}^T\mathbf{e}}{(\xi^2+y^2)^{\frac{3}{2}}}\ d\xi = \frac{1}{4\pi}\frac{\mathbf{y}^T\mathbf{e}}{r}\ \frac{\mathrm{sign}(x)}{r+|x|},\end{aligned}$$

and define, for $x \neq 0$, the vector field $\mathbf{u}_C$ by the equation

$$\mathbf{u}_C(\mathbf{e},x,\mathbf{y}) = \nabla G_1(\mathbf{e},x,\mathbf{y}) = \begin{cases} -\frac{1}{4\pi}\frac{\mathbf{y}^T\mathbf{e}}{r^3} \\ \frac{1}{4\pi}\frac{1}{r}\frac{\mathrm{sign}(x)}{r+|x|}\left[\mathbf{1}-\frac{1}{r}\left(\frac{1}{r}+\frac{1}{r+|x|}\right)\mathbf{y}\mathbf{y}^T\right]\mathbf{e}. \end{cases} \tag{159}$$

We have the following limits:

$$\lim_{\substack{x\to 0\pm \\ \mathbf{y}\neq 0}} \mathbf{u}_C(\mathbf{e},x,\mathbf{y}) = \begin{cases} -\frac{1}{4\pi}\frac{\mathbf{y}^T\mathbf{e}}{y^3} \\ \pm\frac{1}{4\pi}\frac{1}{y^2}\left(\mathbf{1}-2\frac{\mathbf{y}\mathbf{y}^T}{y^2}\right)\mathbf{e}. \end{cases} \tag{160}$$

Using that

$$\begin{aligned}\widehat{G_1}(\mathbf{e},\mathbf{k},x) &= -i\mathbf{k}^T\mathbf{e}\int_x^{\mathrm{sign}(x)\infty}\widehat{G}(\mathbf{k},\xi)\ d\xi \\ &= \frac{1}{2}i\mathbf{k}^T\mathbf{e}\ \frac{1}{k}\int_x^{\mathrm{sign}(x)\infty} e^{-k|\xi|}d\xi = \frac{1}{2}i\mathbf{k}^T\mathbf{e}\ \frac{\mathrm{sign}(x)}{k^2}e^{-k|x|},\end{aligned}$$

we get in Fourier space, for $x \in \mathbf{R} \setminus \{0\}$, that

$$\widehat{\mathbf{u}_C}(\mathbf{e},\mathbf{k},x) = \begin{pmatrix} \partial_x \\ -i\mathbf{k} \end{pmatrix} \widehat{G_1}(\mathbf{e},\mathbf{k},x) = \begin{cases} -\frac{i}{2}\mathbf{k}^T \frac{1}{k} e^{-k|x|}\ \mathbf{e} \\ \frac{1}{2} P_1\ \mathrm{sign}(x) e^{-k|x|}\ \mathbf{e}. \end{cases} \tag{161}$$

The restriction to $x \geq 1$ of the vector field (159) and (161) multiplied by $-2\mathbf{b}$ is one of the terms in the asymptotic expressions (8)–(10) and (114)–(116), respectively. Next, define for $x > 0$ the function H by the equation

$$H(\mathbf{k},x) = \theta(x) e^{-k^2 x},$$

with θ the Heaviside function. For $x > 0$ this is nothing else than the heat Kernel in Fourier space and therefore, for $x > 0$,

$$H(x,\mathbf{y}) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathbf{R}^2} e^{-i\mathbf{k}\cdot\mathbf{y}}\ H(\mathbf{k},x)\ d^2\mathbf{k} = \frac{1}{4\pi x} e^{-\frac{y^2}{4x}}.$$

The vector field

$$\widehat{\mathbf{u}_W}(\mathbf{k},x) = \begin{cases} H(\mathbf{k},x) \\ i\mathbf{k} H(\mathbf{k},x) \end{cases} \tag{162}$$

is divergence free, and for its inverse Fourier transform we have, for $x > 0$,

$$\mathbf{u}_W(x,\mathbf{y}) = \begin{cases} \frac{1}{4\pi x} e^{-\frac{y^2}{4x}} \\ \frac{\mathbf{y}}{8\pi x^2} e^{-\frac{y^2}{4x}}. \end{cases} \tag{163}$$

The restriction to $x \geq 1$ of the vector field (163) and (162) multiplied by c is one of the terms in the asymptotic expressions (8)–(10) and (114)–(116), respectively. Finally, for $x > 0$, let

$$\widehat{\mathbf{u}_V}(\mathbf{e},\mathbf{k},x) = \begin{cases} 0 \\ -P_1 H(\mathbf{k},x)\mathbf{e}. \end{cases} \tag{164}$$

To compute the Fourier transform we define

$$H_1(\mathbf{e},\mathbf{k},x) = \frac{-i\mathbf{k}^T\mathbf{e}}{k^2} H(\mathbf{k},x) = \frac{-i\mathbf{k}^T\mathbf{e}}{k^2} e^{-k^2 x} = \int_x^\infty \left(-i\mathbf{k}^T\mathbf{e}\right)\ e^{-k^2 t} dt,$$

which becomes in direct space

$$H_1(\mathbf{e},x,\mathbf{y}) = -\frac{\mathbf{y}^T\mathbf{e}}{8\pi} \int_x^\infty \frac{1}{\xi^2} e^{-\frac{y^2}{4\xi}}\ d\xi = \frac{\mathbf{y}^T\mathbf{e}}{2\pi} \frac{1}{y^2} \left(e^{-\frac{y^2}{4x}} - 1\right).$$

Therefore,

$$\mathbf{u}_V(\mathbf{e},x,\mathbf{y}) = \begin{cases} 0 \\ \frac{1}{2\pi}\left[\frac{1}{y^2}\left(e^{-\frac{y^2}{4x}} - 1\right)\mathbf{1} - 2\left(\frac{1}{y^2}\left(e^{-\frac{y^2}{4x}} - 1\right) + \frac{1}{4x} e^{-\frac{y^2}{4x}}\right)\frac{\mathbf{y}\mathbf{y}^T}{y^2}\right]\mathbf{e}, \end{cases} \tag{165}$$

we have the limit

$$\lim_{\substack{x\to 0_+\\ \mathbf{y}\neq 0}} \mathbf{u}_V(\mathbf{e},x,\mathbf{y}) = \begin{cases} 0 \\ -\dfrac{1}{2\pi}\dfrac{1}{y^2}\left(\mathbf{1} - 2\dfrac{\mathbf{y}\mathbf{y}^T}{y^2}\right)\mathbf{e}. \end{cases} \tag{166}$$

The restriction to $x \geq 1$ of the vector field (165) and (164) multiplied by $\mathbf{a}$ is one of the terms in the asymptotic expressions (8)–(10) and (114)–(116), respectively.

Finally, as indicated in the introduction, if we replace in (8)–(10) c by $-2d\theta(x)$ and $\mathbf{a}$ by $-2\mathbf{b}\theta(x)$, then we expect (8)–(10) to be the correct asymptotic behavior of a vector field of a stationary fluid flow around a body at large distances not only for $x \to \infty$ but on all curves for which $|x| + |\mathbf{y}| \to \infty$. These asymptotic expressions should therefore be smooth functions away from a neighborhood of the origin. Indeed, since $\mathbf{u}_V = 0$ for $x < 0$, the expressions (8)–(10) are continuous (smooth) at $x = 0$, $\mathbf{y} \neq 0$ if

$$\lim_{\substack{x\to 0_-\\ \mathbf{y}\neq 0}} \mathbf{u}_C(-2\mathbf{b},x,\mathbf{y}) = \lim_{\substack{x\to 0_+\\ \mathbf{y}\neq 0}} \mathbf{u}_V(\mathbf{a},x,\mathbf{y}) + \lim_{\substack{x\to 0_+\\ \mathbf{y}\neq 0}} \mathbf{u}_C(-2\mathbf{b},x,\mathbf{y}),$$

and from (160) and (166) we see that this is only the case if $\psi = \mathbf{a} + 2\mathbf{b} = 0$.

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(accepted: May 13, 2004; published Online First: June 14, 2005)